RE-STRUCTURED SEQUENCES OF LINEAR POSITIVE OPERATORS FOR HIGHER ORDER Lp-APPROXIMATION

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By

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CERTIFICATE

Certified that the work presented in this thesis entitled 'RE-STRUCTURED SEQUENCES OF LINEAR POSITIVE OPERATORS FOR HIGHER ORDER L_p-APPROXIMATION' by Shree Thakur Ashok Kumar Sinha has been carried out under my supervision and that this has not been submitted elsewhere for a degree or diploma.

August, 1981 *(R.K.S. RATHORE)*

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CONTENTS

SYNOPSIS

Following the celebrated Weierstrass approximation theorem much of the classical work in Approximation Theory concentrated around best approximation by algebraic and trigonometric polynomials, rational functions and entire functions of exponential type. Theorems of Jackson, Bernstein and Zygmund type (Natanson (1964) and Timan (1966)) characterized structural properties of functions in terms of their degree of approximation. Such results were obtained both for continuous functions (in sup-norms) and for functions in Lebesgue spaces (in L_p-norms, $1 \leq p < \infty$). The results which determine structural characteristics of functions from their degree of approximation (inverse theorems) mostly made use of Bernstein type inequalities alongwith a telescoping argument; while those which infer the degree of approximation from given structural characteristics of functions (direct theorems) depended on certain linear methods of approximation in their proofs. This, alongwith the studies on Fourier series (Zygmund (1968)), Summability theory and polynomial interpolation, motivated a separate study of linear approximation methods. The fact that the best approximation operators (in sup- and L_p-norms, $p \neq 2$) turn out to be non-linear and hence not so easy to construct seems to be another important reason for the study of linear operators in approximation theory.

An important class of linear operators which has been studied quite extensively from the point of view of approximatio theory consists of convolutions, both on the real line and on the circle group. Some particular operators of this type e.g., the Gauss-Weierstrass integrals led to a study of more general operators of integral type i.e., those defined through an expression of the form

$$L_n(f,t) = \int_J K_n(t,u)f(u)du,$$

the integral being in the Lebesgue sense. Also, operators such as the Bernstein polynomials (originally intended to furnish an extremely elegant proof of the Weierstrass theorem) gave rise to several other classes of operators of summation type. These operators of summation type are defined by expressions of the type

$$L_n(f,x) = \sum_{k \in I} a_{n,k}(x)\, f(x_{k,n}),$$

where the index set I is a finite or infinite subset of the set of natural numbers. If, in the definition of integral type operators, we allow integration in the Lebesgue-Stieltjes sense, the summation type operators may also be expressed in a similar form as the operators of integral type.

Most of the classical linear operators viz., Gauss-Weierstrass integrals, Fejér operators, Jackson operators, de la Vallée Poussin operators and Bernstein polynomials are linea

positive operators. Regarding the convergence of linear positive operators an astonishingly simple set of necessary and sufficient conditions was given by Bohman (1952) and Korovkin (1953). Following them, rates of convergence and inverse and saturation theorems with respect to sup-norm, for particular sequences of linear positive operators have been extensively studied.

Several of the well-known integral type operators may also be used to approximate functions in L_p-norms ($1 \leq p < \infty$). However, for obvious reasons summation type operators as such are not L_p-approximation methods. Nevertheless, several linear positive operators of summation type have been appropriately modified to become L_p-approximation method. The underlying idea behind such a modification is to replace, in the expression for the operator, the function value at a nodal point by an average value (in the sense of integration) of the function in an appropriate neighborhood of the point. The first such modification was made by Kantorovitch (1930) for the case of Bernstein polynomials. Thus with the Bernstein polynomials $B_n(f,t)$ given by

$$B_n(f,t) = \sum_{\nu=0}^{n} p_{n\nu}(t)\, f(\tfrac{\nu}{n}), \left(p_{n\nu}(t) = \binom{n}{\nu} t^{\nu} (1-t)^{n-\nu}\right),$$

the modified polynomials $P_n(f,t)$ are defined by

$$P_n(f,t) = \sum_{\nu=0}^{n} p_{n\nu}(t)\,(n+1) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} f(u)\, du.$$

Approximation by Bernstein-Kantorovitch polynomials in L_p-norms $(1 \leq p < \infty)$ has been studied by Lorentz (1932), Butzer (1952), Hoeffding (1971), Bojanic and Shisha (1975), Ditzian and May (1976), Grundmann (1976), Müller (1976), Maier (1978), Riemenschneider (1978), May (1979) and Becker and Nessel (1979 and 1980). A more detailed description of these works will be given in the first chapter of the thesis. These works establish that the optimal rate of convergence for the Bernstein-Kantorovitch polynomials in L_p-norms is $O(n^{-1})$, which is also the optimal rate of convergence for the corresponding original operators with respect to the sup-norm.

More generally, even though the linear positive operators are conceptually simpler, easier to construct and study, they lack in the rapidity of convergence for sufficiently smooth functions. In the same context a well-known theorem of Korovkin (1960) states that the optimal rate of convergence for any sequence of linear positive polynomial operators is at most $O(n^{-2})$. Thus, if we want to have a better order of approximatio for smootner functions we have to slacken the positivity condition. However, as no general constructional guidelines are available for producing fast analytic approximation methods of a given type, it seems best to start with an appropriate sequence of linear positive operators and then to modify it so as to suit the desired requirements. Three such methods which have been proposed in the literature are :

(a) taking appropriate linear combinations of linear positive operators (Butzer (1953), Rathore (1973) and May (1976));

(b) multiplying the kernel of an integral type linear positive operator by a suitable factor so as to produce appropriate finite oscillations (Stark 1970 and 1972) and Hoff (1970 and 1974)); and

(c) using appropriate linear combinations of iterates of linear positive operators (Micchelli (1973), Agrawal (1979) & Bleimann, Jungeburth and Stark (1979)).

It may be noted that the methods (a), (b) and (c) in the available literature have been studied only with respect to the sup-norms. In this thesis the problems that we study are with respect to the L_p-norms and these are :

(1) Direct, inverse and saturation theorems for linear combinations of certain sequences of linear positive operators in L_p-norms ($1 \leq p < \infty$); and

(2) The possibility of utilizing the classical Newton interpolation polynomials to modify certain sequences of linear positive operators to enhance the rate of approximation in L_p-norms ($1 \leq p < \infty$) and then to study the direct, inverse and saturation theorems for the modified sequences in L_p-norms.

This proposed modification is done as follows. Let $\{L_n\}$ be a sequence of linear positive operators. Then with

$$\Delta^{j}_{n^{-1/2}} f(u) = \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} f(u + \frac{k}{n^{1/2}}),$$

the modified operators $L_{n,m}(f,t)$ of the first type are recursively defined by

$$L_{n,0}(f,t) = L_n(f,t)$$

$$L_{n,m}(f,t) = L_{n,m-1}(f,t)$$

$$+ \frac{n^{m/2}}{m!} L_n((\prod_{j=0}^{m-1}(t-u-\frac{j}{n^{1/2}}) \Delta^{m}_{n^{-1/2}} f(u),t) ,$$

where m is a positive integer.

These modifications are appropriate for functions which are defined over a bounded interval of the real line. For the case of an unbounded interval, however, one of the appropriate modifications is to recursively define

$$L_{n,0}(f,t) = L_n \left(\frac{f(u)}{1+|u-t|^{m_0+2}} , t\right) ,$$

$$L_{n,m}(f,t) = L_{n,m-1}(f,t)$$

$$+ \frac{n^{m/2}}{m!} L_n \left(\frac{\prod_{j=0}^{m-1} (t-u-\frac{j}{n^{1/2}})}{1+|u-t|^{m_0+2}} \Delta^{m}_{n^{-1/2}} f(u),t \right),$$

where the order m of the modification $L_{n,m}(f,t)$ is allowed to be at-most m_0, where m_0 is a positive integer. These modifications are termed as those of the second type.

The sequences of operators which are studied in the thesis under both the categories (1) and (2) are those of the Bernstein-Kantorovitch polynomials $P_n(\cdot,t)$ and the operators $S_n(\cdot,t)$ of exponential type introduced by May (1976).

The thesis is divided into five chapters, I-V. Throughout it the usual linear combinations of order k of Bernstein-Kantorovitch polynomials $P_n(\cdot,t)$ have been denoted by $P_n(\cdot,k,t)$, while those of the exponential type operators $S_n(\cdot,t)$ by $S_n(\cdot,k,t)$. The interpolatory modifications $P_{n,m}(\cdot,t)$ of $P_n(\cdot,t)$ studied in this thesis are of the first type, and the considered interpolatory modifications $S_{n,m}(\cdot,t)$ of $S_n(\cdot,t)$ are of the second type, m denoting the order in either case. The direct, inverse and saturation theorems obtained in this thesis are local in nature. Thus these are in the set up of contracting intervals.

A chapterwise summary of the thesis is as follows :

Chapter I : is of introductory nature containing basic definitions, results and tools for later analysis. In Section 1 we discuss finite differences and Newton's interpolation formula and generalize a certain variational lemma of Lorentz (1966). Section 2 contains some classical results on L_p-spaces including the classical Riesz-Thorin interpolation theorem. Section 3 is a discussion of Steklov means and integral modulii of smoothness and contains the proofs of some of their interrelations and properties in the context of L_p-functions. Section 4 contains

some estimates of Sikkema and Rathore (1976) on a sequence of convolution operators generated by powers of bell shaped functions. These are subsequently to be used in Chapter II for obtaining bounds of adjoint moments of Bernstein-Kantorovitch polynomials. Sections 5 and 6 contain formal definitions of Bernstein-Kantorovitch polynomials, regular exponential type operators, their linear combinations and their interpolatory modifications, in the respective cases, of the first and the second types. The last two Sections contain some basic results and a brief review of the earlier work on these operators.

Chapter II : is a study of the L_p-approximation $(1 \leq p < \infty)$ by linear combinations of Bernstein-Kantorovitch polynomials. It is first shown in Section 1 that the duals of the Bernstein-Kantorovitch polynomials constitute an L_p-approximating sequence. Next some bounds for the moments of these dual operators are obtained. In Section 2 it is shown that for sufficiently smooth functions the linear combinations of Bernstein-Kantorovitch polynomials have a faster rate of convergence in L_p-approximation than that of the Bernstein-Kantorovitch polynomials themselves. In this connection some general bounds for the error in L_p-approximation by linear combinations of Bernstein-Kantorovitch polynomials are obtained. The first of these is in terms of the L_p-norms of derivatives of the function. Here the proof in the case $p > 1$ is made to depend on the Hardy-Littlewood majorant function. The second error bound is expressed in terms

of a higher order integral modulus of smoothness and the proof relies on some properties of Steklov means. Sections 3 and 4 are devoted to inverse and saturation theorems. The Bernstein type inequalities required in the proof of the inverse theorem are obtained through a use of Riesz-Thorin interpolation theorem, whereas for similar results in Chapter IV and V we adopt a more direct method.

Chapter III: is a study of the interpolatory modifications $P_{n,m}(\cdot,t)$ of Bernstein-Kantorovitch polynomials. In Section 1 it is shown that $\{P_{n,m}(\cdot,t)\}$ is an L_p-approximating sequence $(1 \leq p < \infty)$. In Section 2 it is shown that a bound for the error in L_p-approximation by $P_{n,m}(\cdot,t)$ is obtainable essentially in terms of the (m+1)th derivative of the function. The last two sections contain inverse and saturation theorems. In contrast with the saturation order $O(n^{-(k+1)})$ for $P_n(\cdot,k,t)$, the saturation order for $P_{n,m}(\cdot,t)$ is $O(n^{-(m+1)/2})$. Moreover, for the interpolatory modifications $P_{n,m}(\cdot,t)$ the trivial class turns out to be much simpler and it consists of the set of functions which are locally polynomials of degree m.

Chapter IV : studies L_p-approximation $(1 \leq p < \infty)$ by linear combinations $S_n(\cdot,k,t)$ of regular exponential type operator the corresponding sup-norm study of which was made by May (1976). The first result of this chapter is that under the regularity assumptions $\{S_n(\cdot,t)\}$ also becomes an L_p-approximation $(1 \leq p < \infty$ sequence. This is proved in Section 1. Following this in

Section 2 L_p-error estimates of two types viz., one involving the L_p-norms of derivatives of function and the other in terms of the (2k+2)th integral modulus of smoothness of the function are obtained. The inverse theorem for $S_n(\cdot,k,t)$ is proved in Section 3. The saturation theorem is proved in Section 4 and here a use of an asymptotic formula for the duals of regular exponential type operators (Agrawal (1979)) makes the proof of the saturation theorem rather simple.

Chapter V: this last chapter is a study of the second type interpolatory modifications $S_{n,m}(\cdot,t)$ of regular exponential type operators. In Section 1 it is shown that $\{S_{n,m}(\cdot,t)\}$ constitutes an L_p-approximating sequence, where $1 \leq p < \infty$. In Section 2 the L_p-error estimates $(1 \leq p < \infty)$ involving norms of derivatives and those in terms of the (m+1)th integral modulus of smoothness of a function are obtained. The last two sections are devoted to the proofs of inverse and saturation theorems for $S_{n,m}(\cdot,t)$. Here it seems worthmentioning that the proof of the saturation theorem for $S_{n,m}(\cdot,t)$ reveals an interesting fact about the regular exponential type operators. By definition

$$a(n) = \int_A^B W(n,t,u)\,dt,$$

where $W(n,t,u)$ is the kernel of the operators. In the case of the Gauss-Weierstrass integrals and the Post-Widder operators, direct computations show that $a(n) = (1 - \frac{\alpha}{n})^{-1}$, where α is the coefficient of t^2 in $p(t)$, which by definition is a polynomial

of degree two given through

$$\frac{\partial W}{\partial t}(n,t,u) = n\,\frac{(u-t)}{p(t)}\,W(n,t,u).$$

That this is no coincidence follows as a corollary to the proof of the saturation theorem. Thus the relation $a(n) = (1 - \frac{\alpha}{n})^{-1}$ is valid for regular exponential type operators in general.

CHAPTER I

SOME BASIC RESULTS AND PRELIMINARIES

The main theme of this thesis is the study of L_p inverse and saturation theorems for two categories of restructured sequences of linear positive operators which are (i) the usual linear combinations and (ii) an interpolatory modification introduced in this thesis. In the available literature L_p direct, inverse and saturation theorems are known only for the basic sequences of linear positive operators ([9], [10], [16], [18], [22], [25], [29], [32], [39], [43], [44], [46], [48-51], [60] and [70]) rather than their linear combinations etc. In this connection it will follow from the work of this thesis that the apparent difficulties with these theorems for linear combinations etc. could be avoided by a proper use of Steklov means. This may be regarded as the main idea of the work. As various proofs in this thesis draw upon various branches of analysis, this chapter contains several basic results from Numerical Analysis, L_p-Spaces, Best Approximations and other areas which will be used extensively throughout the subsequent chapters. Some of these results are Newton's forward difference interpolation formula, error in this formula, Euler-Maclaurin sum formula, Riesz-Thorin interpolation theorem, properties of Hardy-Littlewood majorant

function, inverse theorem on best approximation by trigonometric polynomials etc. and some deductions from these results. Some of these results are stated in Sections 1 and 2. As Steklov means play a major role in the L_p-approximation of functions, bounds for the derivatives of Steklov means in terms of a corresponding order modulus of smoothness of the function are obtained in Section 3. In Section 4 we state a lemma from [64, p.5] regarding a class of bell-shaped functions. This lemma is to be used in Chapter II to obtain an estimate of the adjoint moments of Bernstein-Kantorovitch polynomials. The linear combinations of linear positive operators have been defined in Section 5. In Section 6 we define interpolatory modifications of linear positive operators. In Sections 7 and 8 we state in brief the work done on approximation using Bernstein-Kantorovitch polynomials and exponential type operators.

Throughout this thesis $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}^0$ denote the set of real numbers, positive integers and nonnegative integers respectively.

1.1 NEWTON'S FINITE DIFFERENCES

Let f be a real valued function over $\mathbb{R}$. The mth ($m \in \mathbb{N}$) forward difference of the function f at the point t, of step length δ, is defined as

$$\Delta_\delta^m f(t) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(t+j\delta).$$

As a convention we write $\Delta_\delta^0 f(t) = f(t)$.

Similarly, the mth backward difference of the function f at the point t, of step length δ, is defined as

$$\nabla_\delta^m f(t) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(t-j\delta),$$

and we put $\nabla_\delta^o f(t) = f(t)$.

Using Newton's forward differences the polynomial $p_m(t)$ of degree m, which interpolates f at the points t_i, $i = 0,1,\ldots,m$, is given by [13 , p. 88]

$$(1.1.1) \qquad p_m(t) = \sum_{j=0}^{m} \{ \frac{1}{j!\ \delta^j} \left(\prod_{i=0}^{j-1} (t-t_i) \right) \Delta_\delta^j f(t_o) \},$$

where $\delta = t_{i+1}-t_i$, $i = 0,1,\ldots,m-1$ and $\prod_{i=0}^{j-1} (t-t_i)$ for $j = 0$ is interpreted as 1.

If a function f is m+1 times differentiable, then by Lagrange's interpolation formula [31 , pp. 44-60] the difference in function f and the polynomial p_m at the point t is given by

$$(1.1.2) \qquad f(t) - p_m(t) = \frac{f^{(m+1)}(\xi)}{(m+1)!} \left(\prod_{i=0}^{m} (t-t_i) \right),$$

where $\xi \in (\min_i \{t_i,t\},\ \max_i \{t_i,t\})$.

Adding (1.1.1) to (1.1.2) one obtains

$$(1.1.3) \quad f(t) = \sum_{j=0}^{m} \{ \frac{1}{j!\ \delta^j} \left(\prod_{i=0}^{j-1} (t-t_i) \right) \Delta_\delta^j f(t_o) \} + \frac{f^{(m+1)}(\xi)}{(m+1)!} \left(\prod_{i=0}^{m} (t-t_i \right.$$

As a particular case of (1.1.3) one obtains for $f(y) = y^r (r \in \mathbb{N})$ and $t = 0$

$$(1.1.4)\quad \sum_{j=0}^{m} \left\{ \frac{(-1)^j}{j!\,\delta^j} \left(\prod_{i=0}^{j-1} t_i\right) \Delta_\delta^j\, t_o^r \right\} = \begin{cases} (-1)^m \left(\prod_{i=0}^{m} t_i\right), & r = m+1, \\ 0 & , r < m+1. \end{cases}$$

We now prove a result which generalizes Lemma 3 of [42 , p. 107]. This is proved by using the fact that $f(t)$ is a polynomial of degree m-1 if and only if $\Delta_\delta^m f(t) \equiv 0$.

Lemma 1.1.1. Let $f(t) \in C[a,b]$. If for each m times continuously differentiable function g which has a compact support contained in (a,b), there holds

$$(1.1.5)\qquad \int_a^b f(t)\, g^{(m)}(t)\, dt = 0,$$

then $f(t)$ is a polynomial of degree m-1.

Proof. In view of the above remark it is sufficient to show that for every sufficiently small $\delta > 0$,

$$\Delta_\delta^{m-1} f(t) = \text{constant},\ t \in [a, b-(m-1)\delta].$$

Otherwise there exist 2m distinct and equidistant points t_i inside (a,b) such that

$$(1.1.6)\quad f(t_{2m}) - \binom{m-1}{1} f(t_{2m-1}) + \cdot\cdot + (-1)^{m-1} f(t_{m+1})$$

$$- \{ f(t_m) - \binom{m-1}{1} f(t_{m-1}) + \cdot\cdot + (-1)^{m-1} f(t_1) \} > 0,$$

say. We choose a small $\delta > 0$ such that for every $i(=1,2,\ldots,2m)$ the intervals $(t_i-\delta, t_i+\delta)$ are disjoint and are contained in (a,b). Next we define a function h as follows :

$$h(t_i) = \text{coefficient of } f(t_i) \text{ in } (1.1.6)\ ,$$

$$h(t_i \pm \delta) = h(a) = h(b) = 0,$$

and $h(t)$ is linear between all these points.

Then we define a function g as the mth iterated indefinite integral of h,

$$\text{i.e.,} \quad g(t) = \int_a^t \cdots \int_a^t h(t)\, dt \ldots dt.$$

It follows from the above construction that g and its first m derivatives vanish at end points a and b. Moreover,

supp $g \subset (a,b)$.

Since $\delta^{-1} \int_{t_i-\delta}^{t_i+\delta} f(t)\, h(t)\, dt \to 2f(t_i)\, h(t_i)$, as $\delta \to 0$, we have by (1.1.6) that $\int_a^b f(t)\, h(t)\, dt > 0$, which contradicts the hypothesis. This completes the proof.

The next lemma [69 , p.107] expresses mth forward difference of function in terms of mth derivative of the function.

Lemma 1.1.2. Let $1 \leq p < \infty$ and $f \in L_p[a,b]$. If f has m derivatives, where $f^{(m-1)} \in$ A.C. $[a,b]$ and $f^{(m)} \in L_p[a,b]$, then

$$(1.1.7) \quad \Delta_\delta^m f(t) = \int_0^\delta \cdots \int_0^\delta f^{(m)}\Big(t + \sum_{i=1}^m y_i\Big) dy_1 \ldots dy_m, \; t \in [a, b-m\delta].$$

1.2 L_p-SPACES AND INTERPOLATION THEOREMS

Let $1 \leq p < \infty$, then $L_p[a,b]$ is defined as the class of all complex valued functions f for which $\int_a^b |f(x)|^p\, dx < \infty$, where integration is taken in the Lebesgue sense. The norm in $L_p[a,b]$ is defined by $||f||_{L_p[a,b]} = (\int_a^b |f(x)|^p\, dx)^{1/p}$,

and the functions equal a.e. are identified. The space $L_\infty[a,b]$ consists of complex valued measurable functions which are essentially bounded and is normed by

$$||f||_{L_\infty[a,b]} = \inf \{M; |f(x)| \leq M \text{ a.e. on } [a,b]\}.$$

The spaces A.C. $[a,b]$ and B.V. $[a,b]$ are defined as the classes of absolutely continuous functions over $[a,b]$ and functions of bounded variation over $[a,b]$ respectively. The class of k times continuously differentiable functions on $\mathbb{R}$ which have a compact support is denoted by C_0^k.

For $f \in L_p[0,a]$, $1 < p < \infty$, the Hardy-Littlewood majorant of f is defined as

$$(1.2.1) \qquad H_f(x) = \sup_{\xi \neq x} \frac{1}{\xi - x} \int_x^\xi f(t)dt, \quad (0 \leq \xi \leq a).$$

The following lemma gives a L_p-bound for the function H_f in terms of f.

Lemma 1.2.1. If $1 < p < \infty$ and $f \in L_p[0,a]$, then the Hardy-Littlewood majorant of f defined by (1.2.1) belongs to $L_p[0,a]$ and

$$(1.2.2) \qquad ||H_f||_{L_p[0,a]} \leq 2^{1/p} \frac{p}{p-1} ||f||_{L_p[0,a]}.$$

The lemma follows from [71 , p. 32] and [68 , p. 5].

The next lemma gives a bound for the intermediate derivatives in terms of the highest derivative and the function in L_p-norm $(1 \leq p < \infty)$. The proofs are given in [53 , p.166] and [28 , p.5].

Lemma 1.2.2. Let $1 \leq p < \infty$, $f \in L_p[a,b]$, $f^{(k)} \in$ A.C. $[a,b]$ and $f^{(k+1)} \in L_p[a,b]$. Then

$$(1.2.3)\quad ||f^{(j)}||_{L_p[a,b]} \leq c_j(||f^{(k+1)}||_{L_p[a,b]} + ||f||_{L_p[a,b]}),$$

$j = 1,2,\ldots,k$, where c_j's are certain constants depending only on j,k,p,a and b.

Next, we state the Riesz-Thorin interpolation theorem [54 , p. 231] in a form which is convenient for our purposes.

Lemma 1.2.3. Let T be a linear operator from $L_j[a,b]$ to itself for $j = 1, \infty$. If for each f in $L_j[a,b]$,

$$||Tf||_{L_j[a,b]} \leq M_j||f||_{L_j[a,b]},$$

where M_1, M_∞ are finite constants, then T maps $L_p[a,b]$ $(1 < p < \infty)$ into itself and moreover, for each $f \in L_p[a,b]$,

$$(1.2.4)\qquad ||Tf||_{L_p[a,b]} \leq M_1^{1/p}M_\infty^{1/q}||f||_{L_p[a,b]},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The following theorem of Alaoglu [27 , p. 424] will be used in the proofs of saturation theorems in which the space would be specialized to B.V. $[a,b]$ or $L_p[a,b]$ $(1 < p < \infty)$.

Lemma 1.2.4. Let X be a Banach space and X^* be its conjugate space. Then the closed unit sphere in X^* is compact in the X-topology of X^*.

Lastly, we state a theorem [61 , p. 102] which tells that the integration by parts in Lebesgue-Stieltjes integral is possible under some restrictive conditions.

Lemma 1.2.5. If f(x) and g(x) are two functions of bounded variation over [a,b], we have

$$(1.2.5) \qquad \int_a^b f\, dg + \int_a^b g\, df = f(b^+)g(b^+)-f(a^-)g(a^-),$$

provided that at each point of [a,b] either one at least of the functions f and g is continuous, or both are regular. (A function f is called regular if $f(x) = \frac{1}{2}(f(x^+)+f(x^-))$ for every $x \in [a,b]$).

2.3 INTEGRAL MODULUS OF SMOOTHNESS AND STEKLOV MEANS

In this section we obtain estimates for derivatives of Steklov means of mth order in terms of a corresponding order integral modulus of smoothness of the function. For particular values of m (= 2,3) these have been proved by Ditzian and May [25 , pp. 739 and 746]. Also, when the function is periodic and belongs to L_p a proof is given in Zygmund [71, p. 117], Achieser [1 , p. 173] and Timan [69 , p.167]. Next, from a given order estimate of the integral modulus of smoothness of order m we deduce smoothness properties of the function. We give two proofs of this. In the first proof we estimate a lower integral modulus of smoothness in terms of a higher order integral modulus of smoothness and then using a result on best approximation by trigonometric polynomials

we complete the proof. The second proof employs a method of induction.

For $f \in L_p[a,b]$ where $1 \le p < \infty$, the integral modulus of smoothness of order m is defined as

$$\omega_m(f,\tau,p,[a,b]) = \sup_{0<\delta\le\tau} \|\Delta_\delta^m f(t)\|_{L_p[a,b-m\delta]}.$$

Let $1 \le p < \infty$, $f \in L_p[a,b]$ and $[a_1,b_1] \subset (a,b)$. Then, for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of mth order corresponding to f is defined as follows :

$$(1.3.1)\quad f_{\eta,m}(u) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left(f(u)+(-1)^{m-1} \Delta^m_{\sum_{i=1}^m u_i} f(u)\right) \times du_1 \ldots du_m,\quad u \in [a_1,b_1].$$

Lemma 1.3.1. Let $1 \le p < \infty$, $[a_1,b_1] \subset (a,b)$ and $f \in L_p[a,b]$. Then, for sufficiently small $\eta > 0$, $f_{\eta,m}$ has derivatives upto order m, the (m-1)th derivative is absolutely continuous over $[a_1,b_1]$ and the mth derivative exists a.e. and belongs to $L_p[a_1,b_1]$. Moreover, there hold :

$$(1.3.2)\quad \|f^{(r)}_{\eta,m}\|_{L_p[a_1,b_1]} \le M_r\, \eta^{-r}\, \omega_r(f,\eta,p,[a,b]),\ r=1,2,\ldots,m;$$

$$(1.3.3)\quad \|f-f_{\eta,m}\|_{L_p[a_1,b_1]} \le M_{m+1}\, \omega_m(f,\eta,p,[a,b]);$$

$$(1.3.4)\quad \|f_{\eta,m}\|_{L_p[a_1,b_1]} \le M_{m+2} \|f\|_{L_p[a,b]};$$

and

$$(1.3.5) \qquad ||f_{n,m}^{(m)}||_{L_p[a_1,b_1]} \leq M_{m+3}\, n^{-m}\, ||f||_{L_p[a,b]} ,$$

where M_i's are certain constants independent of f and n.

Proof. By repeated application of Theorem 18.17 of [30] it follows that $f_{n,m}$ has derivatives upto order m over $[a_1,b_1]$, $f_{n,m}^{(m-1)} \in A.C.[a_1,b_1]$ and the mth derivative exists a.e. and $f_{n,m}^{(m)} \in L_p[a_1,b_1]$.

Writing $h_k(u) = \int_{-n/2}^{n/2} \cdots \int_{-n/2}^{n/2} f(u+k\sum_{i=1}^{m} u_i)du_1 \ldots du_m$, $1 \leq k \leq m$,

by Theorem 18.17 of [30] we have

$$h_k^{(1)}(u) = k^{-1} \int_{-n/2}^{n/2} \cdot\cdot \int_{-n/2}^{n/2} \Delta_{kn}\, f(u-\frac{kn}{2} + k\sum_{i=1}^{m-1} u_i)du_1 \ldots du_{m-1}.$$

A repeated differentiation of the above expression gives

$$(1.3.6) \qquad h_k^{(r)}(u) = k^{-r} \int_{-n/2}^{n/2} \cdot\cdot \int_{-n/2}^{n/2} \Delta_{kn}^{r} f(u-\frac{rkn}{2} + k\sum_{i=1}^{m-r} u_i)du_1 \cdot\cdot du_{m-r},$$

and

$$(1 \leq r \leq m-1)$$

$$(1.3.7) \qquad h_k^{(m)}(u) = k^{-m} \Delta_{kn}^{m}\, f(u - \frac{kmn}{2}) \text{ a.e.}$$

From (1.3.1) and the definition of $\Delta_\delta^m f(u)$

$$(1.3.8) \qquad f_{n,m}(u) = \frac{(-1)^{m-1}}{n^m} \left\{ \sum_{k=1}^{m} \binom{m}{k} (-1)^{m-k}\, h_k(u) \right\} .$$

From this one obtains

$$(1.3.9) \qquad f_{n,m}^{(r)}(u) = \frac{(-1)^{m-1}}{n^m} \left\{ \sum_{k=1}^{m} \binom{m}{k} (-1)^{m-k}\, h_k^{(r)}(u) \right\} .$$

Using Jensen's inequality repeatedly we obtain from (1.3.6)

$$(1.3.10) \quad |h_k^{(r)}(u)|^p \leq k^{-rp} \, \eta^{(m-r)(p-1)}$$

$$\times \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} |\Delta_{k\eta}^{r} \, f(u-\tfrac{rk\eta}{2} + k \sum_{i=1}^{m-r} u_i)|^p \, du_1 \ldots du_{m-r}.$$

Next, using Fubini's theorem m-r times to interchange integrals we obtain

$$\int_{a_1}^{b_1} |h_k^{(r)}(u)|^p \, du \leq k^{-rp} \, \eta^{(m-r)(p-1)} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \int_{a_1}^{b_1}$$

$$\times \{ |\Delta_{k\eta}^{r} f(u - \tfrac{rk\eta}{2} + k \sum_{i=1}^{m-r} u_i)|^p \} \, du \, du_1 \ldots du_{m-r}$$

$$\leq M \, \eta^{(m-r)p} \, (\omega_r(f, k\eta, p, [a_1, b_1]))^p$$

$$(1.3.11) \qquad \leq M_r^p \, \eta^{(m-r)p} \, (\omega_r(f, \eta, p, [a_1, b_1]))^p.$$

Thus, (1.3.2) follows from (1.3.9) and (1.3.11). Proceeding similarly we obtain estimates (1.3.3) to (1.3.5).

Theorem 1.3.2. Let $1 \leq p < \infty$, $f \in L_p[a,b]$ and there hold

$$\omega_m(f, \tau, p, [a,b]) = O(\tau^{r+\alpha}), \quad (\tau \to 0),$$

where $m, r \in \mathbb{N}$ and $0 < \alpha < 1$.

Then $f(x)$ coincides a.e. on $[c,d] \subset (a,b)$ with a function $F(x)$ possessing an absolutely continuous derivative $F^{(r-1)}(x)$, the rth derivative $F^{(r)}(x) \in L_p[c,d]$, and there holds

$$\omega(F^{(r)}, \tau, p, [c,d]) = O(\tau^{\alpha}), \quad (\tau \to 0).$$

Proof. First method : We first prove a lemma which gives a bound for a lower order integral modulus of smoothness in terms

of a higher order integral modulus of smoothness.

Lemma 1.3.3. Let $1 \leq p < \infty$ and $f \in L_p[a,b]$. Then

$$(1.3.12) \quad \omega_k(f,\tau,p,[a,b]) \leq M_k \tau^k \{||f||_{L_p[a,b]} + \int_\tau^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} du\},$$

where the constant M_k does not depend on f.

Proof. Since the proof is similar to 3.3(11) of [69, p. 108], we only sketch it. We have

$$||\Delta_{2h}^k f(x)-2^k \Delta_h^k f(x)||_{L_p[a,b-2kh]}$$

$$= ||\sum_{\nu=0}^{k} \binom{k}{\nu} \{\Delta_h^k f(x+\nu h)-\Delta_h^k f(x)\}||_{L_p[a,b-2kh]}$$

$$\leq \sum_{\nu=1}^{k} \binom{k}{\nu} \{\sum_{\mu=0}^{\nu-1} ||\Delta_h^{k+1} f(x+\mu h)||_{L_p[a,b-2kh]}\}$$

$$\leq k\, 2^{k-1} \omega_{k+1}(f,h,p,[a,b]).$$

Putting here, successively, $h = 2^m \epsilon$ $(m = 0,1,\ldots,r-1)$ and taking $0 \leq \epsilon \leq \tau$, we obtain a system of inequalities of the form

$$||\Delta_{2^{m+1}\epsilon}^k f(x)-2^k \Delta_{2^m\epsilon}^k f(x)||_{L_p[a,b-2kh]}$$

$$\leq k\, 2^{k-1} \omega_{k+1}(f,2^m\epsilon,p,[a,b]).$$

Multiplying both sides by $2^{-(m+1)k}$ and summing for $m = 0,1,\ldots,r-1$ and using the fact that $\omega_{k+1}(f,u,p,[a,b])$ is nondecreasing with u we get

$$||2^{-rk}\,\Delta^k_{2^r\epsilon}\, f(x)-\Delta^k_\epsilon\, f(x)||_{L_p[a,b-2^r\tau k]}$$

$$\leq k^2\,\tau^k \int_\tau^{2^r\tau} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\,du,$$

which implies that

$$(1.3.13)\quad ||\Delta^k_\epsilon\, f(x)||_{L_p[a,b-2^r\,\tau k]} \leq 2^{-(r-1)k}||f||_{L_p[a,b]}$$

$$+\ k^2\,\tau^k \int_\tau^{2^r\tau} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\,du.$$

Proceeding as in above, with $f(x)$ replaced by $\phi(x) = f(a+b-x)$, one obtains

$$(1.3.14)\quad ||\Delta^k_\epsilon\,\phi(x)||_{L_p[a,b-2^r\tau k]} \leq 2^{-(r-1)k}||f||_{L_p[a,b]}$$

$$+\ k^2\tau^k \int_\tau^{2^r\tau} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\,du.$$

Now,

$$(1.3.15)\quad \left(\int_a^{b-k\epsilon} |\Delta^k_\epsilon\, f(x)|^p\, dx\right)^{1/p} \leq \left(\int_a^{\frac{a+b}{2}} |\Delta^k_\epsilon\, f(x)|^p dx\right)^{1/p}$$

$$+\ \left(\int_{\frac{a+b}{2}}^{b-k\epsilon} |\Delta^k_\epsilon\, f(x)|^p\, dx\right)^{1/p} = J_1+J_2, \text{ say.}$$

We have

$$J_2 = \left(\int_{\frac{a+b}{2}}^{b-k\epsilon} |\Delta^k_\epsilon\, f(x)|^p\, dx\right)^{1/p} = \left(\int_{a+k\epsilon}^{\frac{a+b}{2}} |\nabla^k_\epsilon\, f(a+b-y)|^p\, dy\right)^{1/p}$$

$$= \left(\int_{a+k\epsilon}^{\frac{a+b}{2}} |\nabla^k_\epsilon\,\phi(y)|^p\, dy\right)^{1/p} = \left(\int_{a+k\epsilon}^{\frac{a+b}{2}} |\Delta^k_\epsilon\,\phi(x-k\epsilon)|^p\, dx\right)^{1/p}$$

$$(1.3.16)\qquad = \left(\int_a^{\frac{a+b}{2}-k\epsilon} |\Delta^k_\epsilon\,\phi(y)|^p\, dy\right)^{1/p}.$$

Hence, from (1.3.13), (1.3.14), (1.3.15) and (1.3.16), it follows that

(1.3.17) $$||\Delta_\epsilon^k f(x)||_{L_p[a,b-k\epsilon]} \leq 2\{2^{-(r-1)k}||f||_{L_p[a,b]} + k^2\tau^k \int_\tau^{2^r\tau} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\,du\}.$$

We choose integer $r = r(\tau)$ such that

$$\frac{b-a}{4} \leq 2^r \tau k \leq \frac{b-a}{2}.$$

This gives, by (1.3.17), that

(1.3.18) $$||\Delta_\epsilon^k f(x)||_{L_p[a,b-k\epsilon]} \leq M_k \tau^k\{||f||_{L_p[a,b]} + \int_\tau^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}}\,du\}.$$

Since $\epsilon \leq \tau$ is arbitrary the proof follows from (1.3.18).

<u>Corollary 1.3.4.</u> Let $k < m$ be a positive integer and let

$$\omega_m(f,\tau,p,[a,b]) = O(\tau^\beta), \quad (\tau \to 0).$$

Then there holds as $\tau \to 0$

$$\omega_k(f,\tau,p,[a,b]) = \begin{cases} O(\tau^k), & \beta > k, \\ O(\tau^k|\ln \tau|), & \beta = k, \\ O(\tau^\beta), & \beta < k. \end{cases}$$

Now by using following lemma, on best approximation by trigonometric polynomials([69, Theorem 6.1.4, p. 339]), in conjunction with Corollary 1.3.4, we will complete a proof of the theorem.

Lemma 1.3.5. If the sequence of best approximations $E_n^*(f)_{L_p}$ of the periodic function $f \in L_p[0,2\pi]$ $(1 \leq p \leq \infty)$ for a certain positive integer r and $\alpha > 0$ satisfies the relation

$$E_n^*(f)_{L_p} = O(\frac{1}{n^{r+\alpha}}), \quad (n \to \infty),$$

then a.e. on $(-\infty,\infty)$ the function $f(x)$ coincides with a function possessing an absolutely continuous derivative $f^{(r-1)}(x)$ and a rth derivative $f^{(r)}(x)$ which belongs to L_p over a period and for any integer k, as τ goes to zero, there holds

$$\omega_k(f^{(r)}, \tau)_{L_p} = \begin{cases} O(\tau^{\alpha}), & \alpha < k, \\ O(\tau^k |\ln \tau|), & \alpha = k, \\ O(\tau^k), & \alpha > k. \end{cases}$$

Proof of the theorem. We choose points a_1, b_1 such that $a < a_1 < b_1 < b$. Let $g \in C_0^m$ with $\operatorname{supp} g \subset (a_1,b_1)$. Then it follows from Corollary 1.3.4 and the relation

$$\Delta_\tau^m(fg)(x) = \sum_{j=0}^{m} \binom{m}{j} (\Delta_\tau^{m-j} f(x)) \Delta_\tau^j g(x+(m-j))$$

that

$$\omega_m(fg,\tau,p,[a,b]) = O(\tau^{r+\alpha}), \ (\tau \to 0).$$

Now, we extend the function fg periodically with period (b−a) over $\mathbb{R}$. Let G denote the extended function. We define another function F with the help of G which is 2π periodic and for which $\omega_m(F,\tau,p,[0,2\pi]) = O(\tau^{r+\alpha})$, $(\tau \to 0)$.

The function F is defined as follows :

$$F(x) = G(a + \frac{b-a}{2\pi} x).$$

Then by (5.11.1) of [69, p. 326] we have

$$(1.3.19)\qquad E_n^*(F)_{L_p} \le c_m\, \omega_m(F, \frac{1}{n+1}, p, [0,2\pi]) \le \frac{c_m'}{n^{r+\alpha}} .$$

Now, choosing a particular $g \in C_0^m$ such that $g(x) = 1$ for $x \in [c,d]$, the theorem follows from (1.3.19) and Lemma 1.3.5.

Second Method. The proof makes use of another theorem on best approximation by trigonometric polynomials ([69, Theorem 6.1.2, p. 335]) which we state below :

Lemma 1.3.6. If the sequence of best approximations $E_n^*(f)_{L_p}$ of the periodic function $f \in L_p[0,2\pi]$ $(1 \le p \le \infty)$ satisfies, for a certain $\alpha > 0$, the relation

$$E_n^*(f)_{L_p} = O(\frac{1}{n^\alpha}), \quad (n \to \infty),$$

then for any integer k, there holds for all sufficiently small values of τ

$$\omega_k(f,\tau)_{L_p} = \begin{cases} O(\tau^\alpha), & \alpha < k, \\ O(\tau^k|\ln \tau|), & \alpha = k, \\ O(\tau^k), & \alpha > k. \end{cases}$$

Proof of the theorem. We choose a $g \in C_0^m$ with supp $g \subset (a,b)$ and $g(x) = 1$ for $x \in [c,d]$. Since

$$(1.3.20)\qquad \Delta_\tau^m(fg)(x) = \sum_{j=0}^{m} \binom{m}{j} (\Delta_\tau^{m-j} f(x))(\Delta_\tau^j\, g(x+(m-j)\tau)),$$

it follows from the hypothesis on f and g that

$$\omega_m(fg,\tau,p,[a,b]) = O(\tau^{r+\alpha}) + O(\tau), \ (\tau \to 0)$$
$$= O(\tau^{\beta}), \ (\tau \to 0),$$

for any $\beta < 1$.

Proceeding as in the earlier proof, it follows from Lemma 1.3.6 that $\omega_1(fg,\tau,p,[a,b]) = O(\tau^{\beta})$. Since $g(x) = 1$ for $x \in [c,d]$, we obtain

$$(1.3.21) \qquad \omega_1(f,\tau,p,[c,d]) = O(\tau^{\beta}), \quad (\tau \to 0).$$

Next we choose points a_1, b_1 and another $g \in C_0^m$ with supp $g \subset (a_1,b_1)$ and $g(x) = 1$ for $x \in [c,d]$. Since (1.3.21) holds for every subinterval $[c,d] \subset (a,b)$, we have by (1.3.20) and (1.3.21) that $\omega_m(fg,\tau,p,[a_1,b_1]) = O(\tau^{1+\beta})$, $(\tau \to 0)$. Proceeding as in the proof of the first method we obtain that $f \in$ A.C. $[c,d]$, $f' \in L_p[c,d]$ and $\omega(f',\tau,p,[c,d]) = O(\tau^{\beta})$. Continuing in this manner we get the result.

Next we state a lemma ([12,p.696], [8, p.100]) which will prove useful in the proofs of inverse theorems.

Lemma 1.3.7. Let Ω be a monotonically increasing function on $[0,a]$. Further, let for some $0 < \alpha < r$ and all $\tau, \eta \in (0,a)$ there hold

$$\Omega(\tau) \le M\{\eta^{\alpha} + (\tfrac{\tau}{\eta})^{r}\,\Omega(\tau)\}.$$

Then $\qquad \Omega(\tau) = O(\tau^{\alpha}), \ (\tau \to 0).$

1.4 CONVOLUTIONS WITH POWERS OF BELL-SHAPED FUNCTIONS

We state a lemma from [64, Lemma 3, p. 5] on asymptotic estimates for absolute moments of convolution operators generated by powers of bell-shaped functions β which will be used to obtain bounds for the adjoint moments of Bernstein-Kantorovitch polynomials. The bell-shaped functions are defined as follows.

A function $\beta \in L_1(\mathbb{R})$ is said to be *bell-shaped* if

(i) $\beta(t) \geq 0$ for all $t \in \mathbb{R}$,

(ii) $\beta(t)$ is continuous at $t = 0$ and $\beta(0) = 1$,

and

(iii) $\sup_{|t| > \delta} \beta(t) < 1$ for each $\delta > 0$.

With $\beta(t)$ a bell-shaped function the convolution operator $U_n(f,x)$ is defined as

$$U_n(f,x) = \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} f(t)\, \beta^n(t-x)\, dt$$

for those functions for which the integral is defined and $\alpha(n) = \int_{-\infty}^{\infty} \beta^n(t)\, dt$.

Lemma 1.4.1. Let $\alpha > 0$, β be a bell-shaped function and for some $m \geq 1$, $|t|^\alpha \beta^m(t) \in L_1(\mathbb{R})$. Then, if $\beta''(0)$ exists and is non-zero,

$$(1.4.1) \quad \lim_{n \to \infty} n^{\frac{\alpha+1}{2}} \int_{-\infty}^{\infty} |t|^\alpha \beta^n(t) dt = \Gamma\left(\frac{\alpha+1}{2}\right)\left(\frac{-2}{\beta''(0)}\right)^{\frac{\alpha+1}{2}},$$

where $\Gamma(t)$ is the Gamma function.

1.5 LINEAR COMBINATIONS OF LINEAR POSITIVE OPERATORS

Given a sequence $\{L_n(\cdot,t)\}$ of linear positive operators, following Rathore [55], we define the linear combinations $L_n(\cdot,k,t)$ as follows :

Let $d_o, d_1, \ldots, d_k$ be any k+1 distinct positive numbers such that $L_{d_i n}$'s are meaningful. Then

$$(1.5.1) \qquad L_n(\cdot,k,t) = \frac{1}{\Delta_k} \begin{vmatrix} L_{d_o n}(\cdot,t) & d_o^{-1} & \ldots & d_o^{-k} \\ L_{d_1 n}(\cdot,t) & d_1^{-1} & \ldots & d_1^{-k} \\ \vdots & \vdots & \ldots & \vdots \\ L_{d_k n}(\cdot,t) & d_k^{-1} & \ldots & d_k^{-k} \end{vmatrix},$$

where Δ_k is the determinant obtained after replacing the entries of the first column by 1.

This combination can also be written as (see May [45])

$$(1.5.2) \qquad L_n(\cdot,k,t) = \sum_{j=0}^{k} c(j,k)\, L_{d_j n}(\cdot,t),$$

where $c(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}$, $k \neq 0$, and $c(0,0) = 1$.

Since the linear combinations of several sequences of linear positive operators give an improved order of approximation for sufficiently smooth functions, their convergence in C-norm has been studied quite extensively. To mention a few we refer the works of Butzer [17], Rathore [55], May [45], Agrawal [2] and Ditzian [23].

In this thesis we study L_p-approximation by linear combinations of Bernstein-Kantorovitch polynomials $P_n(\cdot,t)$ and regular exponential type operators $S_n(\cdot,t)$. The operators $P_n(\cdot,t)$ and $S_n(\cdot,t)$ are defined as follows.

Bernstein-Kantorovitch polynomials are a modification of Bernstein polynomials suggested by Kantorovitch [34] for functions belonging to $L_p[0,1]$:

$$P_n(f,t) = (n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} f(u)du\}.$$

Writing $K(n,t,u) = (n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t)\, \chi_{n\nu}(u)\}$,

where $\chi_{n\nu}(u)$ is the characteristic function of the intervals $[\frac{\nu}{n+1}, \frac{\nu+1}{n+1})$ for $\nu = 0,1,\ldots,n-1$, and of $[\frac{n}{n+1}, 1]$ for $\nu = n$, we can write

$$(1.5.3) \qquad P_n(f,t) = \int_0^1 K(n,t,u)\, f(u)\, du.$$

The operators $S_n(f,t)$ ([45]) defined by

$$(1.5.4) \qquad S_n(f,t) = \int_A^B W(n,t,u)\, f(u)\, du,$$

where $W(n,t,u) \geq 0$ is a distributional kernel $(-\infty \leq A < B \leq \infty)$, are said to be of exponential type if

$$(1.5.5) \qquad \int_A^B W(n,t,u)\, du = 1,\ t \in (A,B),$$

and

$$(1.5.6) \qquad \frac{\partial}{\partial t} W(n,t,u) = \frac{n}{p(t)} (u-t)\, W(n,t,u),\ u,t \in (A,B),$$

where $p(t)$ is a polynomial of degree ≤ 2, $p(t) > 0$ for $t \in (A,B)$.

It is further assumed that the range of S_n is contained in $C^\infty(A,B)$ and there holds for $k \in \mathbb{N}$

(1.5.7) $$\frac{d^k}{dt^k} S_n(f,t) = \int_A^B \left(\frac{\partial^k}{\partial t^k} W(n,t,u)\right) f(u)\, du.$$

The operators $S_n(\cdot,t)$ of exponential type are called regular if $W(n,t,u)$ are measurable over $(A,B) \times (A,B)$ and the following conditions are satisfied.

(1.5.8) $$\int_A^B W(n,t,u)dt = a(n), \quad u \in (A,B),$$

where $a(n)$ is a rational function of n satisfying

(1.5.9) $$\lim_{n \to \infty} a(n) = 1,$$

and for each fixed $u \in (A,B)$ and $m \in \mathbb{N}^0$,

(1.5.10) $t^m\, p(t)\, W(n,t,u) \to 0$ as $t \to A$ or $t \to B$, for all sufficiently large values of n.

1.6 INTERPOLATORY MODIFICATIONS OF LINEAR POSITIVE OPERATORS

In Chapters III and V, respectively, we shall study L_p-approximation ($1 \le p < \infty$) of functions by interpolatory modifications of Bernstein-Kantorovitch polynomials $P_n(\cdot,t)$ and regular exponential type operators $S_n(\cdot,t)$. The operators $P_n(\cdot,t)$ and $S_n(\cdot,t)$ are modified by making use of classical Newton interpolation polynomials. This is accomplished by replacing function value $f(u)$ at point 'u' by Newton interpolation polynomial of mth degree based at nodes $u, u+\frac{1}{n^{1/2}}, \ldots, u+\frac{m}{n^{1/2}}$.

For $f \in L_p[0,1]$, where $1 \leq p < \infty$, interpolatory modification $P_{n,m}(f,t)$ of order m of Bernstein-Kantorovitch polynomials $P_n(f,t)$ is defined as follows :

(1.6.1) $P_{n,m}(f,t)$

$$= \int_0^1 K(n,t,u) \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta^j f(u) \} du,$$

where $\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})$ for $j = 0$ is interpreted as 1 and $f(u)$ is zero when $u > 1$. Here and in the sequel Δ denotes $\Delta_{n^{-1/2}}$.

And for $f \in L_p[A,B]$, $1 \leq p < \infty$, interpolatory modification $S_{n,m}(f,t)$, of order m, of regular exponential type operators $S_n(f,t)$ is defined as :

(1.6.2) $S_{n,m}(f,t)$

$$= \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_0+2}} \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta^j f(u) \} du,$$

where $m \leq m_0$.

1.7 A RESUME OF BERNSTEIN-KANTOROVITCH POLYNOMIALS

L_p-approximation by Bernstein-Kantorovitch polynomials has been recently studied by Hoeffding [32], Bojanic and Shisha [16], Grundmann [29], Müller [48], Ditzian and May [25], Maier [43,44], Riemenschneider [60], May [46] and Becker and Nessel [9,10].

Ditzian and May [25] have proved local direct, inverse and saturation theorems in L_p-norm over contracting subintervals. The inverse theorem is as follows :

"Let $0 < \alpha < 2$, $1 \leq p < \infty$ and $f \in L_p[0,1]$. Then for $a < a_1 < b_1 < b$, $||P_n(f,t)-f(t)||_{L_p[a,b]} = O(n^{-\alpha/2})$, $(n \to \infty)$

implies that $\omega_2(f,\tau,p,[a_1,b_1]) = O(\tau^\alpha)$, $(\tau \to 0)$".

When $\alpha = 2$, the following is the saturation theorem :

$$"||P_n(f,t)-f(t)||_{L_p[a,b]} = O(n^{-1}), \quad (n \to \infty),$$

implies that f coincides a.e. on $[a,b]$ with a function F such that $F' \in A.C.[a,b]$ and $F^{(2)} \in L_p[a,b]$ for $p > 1$ and $F' \in B.V.[a,b]$ for $p = 1$ (We would remark here that this result is striking in view of the conclusion being valid with respect to the whole interval $[a,b]$. However, in this context the proof given in [25] is rather sketchy and the details have not been given).

And $||P_n(f,t) - f(t)||_{L_p[a,b]} = o(n^{-1})$, $(n \to \infty)$

implies that f coincides a.e. on $[a,b]$ with a function F satisfying for some constant c

$$t(1-t)\,F'(t) = c \text{ for } t \in [a,b]" .$$

Grundmann [29] and Müller [48] have obtained bounds for the error in L_p-approximation, respectively for the cases $p = 1$ and $p > 1$, in terms of first order integral modulus of smoothness of the function. For $f \in L_p[0,1]$ they have shown that

$$||P_n(f,t)-f(t)||_{L_p[0,1]} \leq M_p \, \omega_1(f,n^{-1/2},p,[0,1]),$$

$M_p = 4$ if $p = 1$. Bojanic and Shisha [16] have obtained the above estimate in weighted L_1 norm. For differentiable functions bounds for error in approximation have been given by Hoeffding [32] and Müller [48] respectively, for the cases $p = 1$ and $p > 1$ as follows :

$$||P_n(f)-f||_{L_p[0,1]} \leq \frac{M}{n^{1/2}} J(f),$$

where $J(f) = \int_0^1 (x(1-x))^{1/2} |df(x)|$ when $p = 1$ and

$J(f) = ||f'||_{L_p[0,1]}$ if $p > 1$.

Maier [43] has proved the following global saturation theorem in $L_1[0,1]$ norm. This has been extended to $L_p[0,1]$, where $1 < p < \infty$ by Riemenschneider [60].

Let $S = \{f; f(x) \doteq k + \int_\xi^x \frac{h(t)}{t(1-t)} dt,\ \xi \in (0,1),\ k \in \mathbb{R}$ and $h \in B.V.[0,1]$ satisfying $h(0)=h(1)=0\}$.

Then $||P_n(f,t) - f(t)||_{L_1[0,1]} = O(n^{-1})$ iff $f \in S$,

and $||P_n(f,t) - f(t)||_{L_1[0,1]} = o(n^{-1})$ iff f is constant a.e.

Becker and Nessel [9] have characterized the saturation class S of Bernstein-Kantorovitch polynomials. They have proved that $f \in S$ is equivalent to

$$||\phi\Delta_h^2 F||_{BV+C[h,1-h]} = O(h^2),\ (h \to 0^+)$$

or, $$||\Delta_h^B F||_{BV+C(I_h)} = O(h^2),\ (h \to 0^+),$$

where $\phi(x) = x(1-x)$, $F(x) = \int_0^x f(x)dt$, $\Delta_h^B F(x) =$

$$= x\int_0^h \int_{(1+h)x-t}^{(1+h-t)x} f'(s)ds\, dt,\ I_h = [h/1+h, 1/1+h],$$

$$||\phi\Delta_h^2 F||_{BV+C[h,1-h]} = ||\phi\Delta_h^2 F||_{BV} + ||\phi\Delta_h^2 F||_{C[h,1-h]}.$$

May [46] proved a more natural global saturation theorem alongwith a correction term, in weighted $L_p[0,1]$ norm, where $1 \leq p < \infty$. The presence of correction term renders the saturation class more natural. The theorem states that

$$||\psi_n^{-1}(t)\{P_n(f,t)-f(t) - \phi_n(t)\, P_n'(f,t)\}||_{L_p[0,1]} = O(1)$$

iff $f' \in$ A.C. $[0,1]$ and $f^{(2)} \in L_p[0,1]$ for $1 < p < \infty$, or $f \in$ A.C. $[0,1]$ and $f' \in$ B.V. $[0,1]$ for $p = 1$.

And, $$||\psi_n^{-1}(t)\{P_n(f,t)-f(t) - \phi_n(t)P_n'(f,t)\}||_{L_p[0,1]} = o(1)$$

iff f is linear, where $\phi_n(t) = \frac{1-2t}{2n}$ and $\psi_n(t) = P_n(\frac{(u-t)^2}{2}, t)$.

Some of the auxiliary results obtained in [25], [40] and [46] will be used in Chapters II and III. We state them as lemmas. These lemmas are about moment formulae of Bernstein polynomials, Bernstein-Kantorovitch polynomials, derivatives of $p_{n\nu}(t)$ etc.

Lemma 1.7.1. ([40, Theorem 1.5.1]) Let $s \in \mathbb{N}$, then $T_{n,s}(t)$ defined as

$$T_{n,s}(t) = \sum_{\nu=0}^{n} (\nu - nt)^s \, p_{n\nu}(t)$$

is a polynomial in t and n; in t of degree $\leq s$, in n of degree $[\frac{s}{2}]$. $T_{n,2s}(t)$ depends only on $t(1-t)$ and the coefficient of n^s is $\frac{(2s)!}{2^s \, s!}(t(1-t))^s$; $T_{n,2s+1}(t)$ is a polynomial in $t(1-t)$, multiplied by $(1-2t)$.

The following lemma ([46, Lemma 2.1, p. 322]) expresses moments of Bernstein-Kantorovitch polynomials in terms of moments of Bernstein polynomials.

Lemma 1.7.2. Let $m \in \mathbb{N}$ and $p(t) = t(1-t)$. Then

$$P_n((u-t)^m,t) = \frac{n+1}{(m+1)\,p(t)} \, B_{n+1}((u-t)^{m+2},t). \tag{1.7.1}$$

The following corollary follows from Lemmas 1.7.1 and 1.7.2.

Corollary 1.7.3. For $m \in \mathbb{N}$ there holds

$$P_n((u-t)^m,t) = \frac{1}{(n+1)^{m+1}} \, Q(n+1,t), \tag{1.7.2}$$

where $Q(n+1,t)$ is a polynomial in $(n+1)$ of degree $[\frac{m+2}{2}]$, and in t of degree $\leq m$.

The derivatives of $p_{n\nu}(t)$ are given by lemma of Lorentz [40].

Lemma 1.7.4. Let $k \in \mathbb{N}$. Then there exist polynomials $q_{ij}^{(k)}(t)$ which do not depend on ν or n, such that

$$(1.7.3)\quad \frac{d^k}{dt^k}\, t^\nu(1-t)^{n-\nu} = \sum_{i,j} n^i(\nu-nt)^j q_{ij}^{(k)}(t)t^\nu(1-t)^{n-\nu}\, T^{-k},$$

where i,j vary over $\mathbb{N}^0$ and satisfy $2i+j \leq k$ and $T(t) = t(1-t)$.

The presence of the factor $(T(t))^k$ in the denominator does not cause any problem for local inverse theorems as it is bounded away from zero over compact subsets of $(0,1)$. However, in the proofs of inverse theorems of a global nature a different type of bound is required (see e.g., [3], [4], [6], [7] and [23]).

The following simple lemma gives a bound for the absolute moments of a general linear positive operator $L_n(f,t)$

$$L_n(f,t) = \int_A^B W(n,t,u)\, f(u)\, du, \quad (t \in I)$$

which maps 1 to itself.

Lemma 1.7.5. If for every $m \in \mathbb{N}$ and all t belonging to some compact subset K of I there holds

$$L_n((u-t)^{2m},t) = O(n^{-m}), \qquad (n \to \infty),$$

then for every positive number r and all $t \in K$

$$(1.7.4)\quad L_n(|u-t|^r,t) = O(n^{-r/2}), \quad (n \to \infty).$$

Proof. Let s be an even integer $> r$. Then using Holder's inequality

$$\int_A^B W(n,t,u)\, |u-t|^r\, du \leq$$

$$\leq \{\int_A^B W(n,t,u)|u-t|^s\, du\}^{r/s} \{\int_A^B W(n,t,u)du\}^{1-\frac{r}{s}}.$$

This gives by assumed moment estimates for integer s that

$$\int_A^B W(n,t,u)\ |u-t|^r\ du \leq M_r\ n^{-r/2}.$$

As a corollary to above lemma we obtain moment estimates for Bernstein polynomials and Bernstein-Kantorovitch polynomials.

Corollary 1.7.6. Let r be a positive number. Then

$$|T_{nr}(t)| \leq M\ n^{-r/2}, \tag{1.7.5}$$

the constant M being independent of t and n.

Corollary 1.7.7. Let r be a positive number. Then

$$P_n(|u-t|^r, t) \leq M\ n^{-r/2}, \tag{1.7.6}$$

the constant M being independent of t and n.

The following results ([25, pp. 734-735]) show that the operators $P_n(.,t)$ are L_p-bounded and that in the L_p-approximation contribution from outside of a neighbourhood of the interval is arbitrarily small.

Lemma 1.7.8. Let $1 \leq p < \infty$ and $f \in L_p[0,1]$. Then

$$||P_n(f,t)||_{L_p[0,1]} \leq ||f||_{L_p[0,1]}. \tag{1.7.7}$$

Lemma 1.7.9. Let $1 \leq p < \infty$ and $f \in L_p[0,1]$. Then for $i \in \mathbb{N}^0$, $[a_1,b_1] \subset (a,b)$ and with $\chi(u)$ as the characteristic

function of the interval $[a,b]$, there holds for any fixed positive number ℓ

(1.7.8) $$\|P_n(f(u)(u-t)^i(1-\chi(u)),t)\|_{L_p[a_1,b_1]}$$

$$= O(n^{-\ell})\,\|f\|_{L_p[0,1]},\ (n \to \infty).$$

The proof follows in the manner of the proof of Lemma 2.3 of [25, p. 735].

The following Euler-Maclaurin sum formula ([13, p.275]) will be required in the proof of saturation theorem for interpolatory modifications of Bernstein-Kantorovitch polynomials.

Lemma 1.7.10. Let function $f(x) \in C[0,1]$ have $2k$ $(2k+1)$ derivatives on $(0,1)$ where $f^{(2k)}(f^{(2k+1)}) \in A.C.[0,1]$. Then

(1.7.9) $$\sum_{j=0}^{n} f(\tfrac{j}{n}) = n\int_0^1 f(x)\,dx + \tfrac{1}{2}(f(0)+f(1))$$

$$+ n\sum_{r=1}^{k} \frac{B_{2r}}{(2r)!\,n^{2r}}\,(f^{(2r-1)}(1)-f^{(2r-1)}(0))-nR,$$

where $$R = -\frac{1}{n^{2k+2}}\{\sum_{r=0}^{n-1}\int_0^1 P_{2k+1}(t)\,f^{(2k+1)}(\tfrac{t+r}{n})\,dt\}$$

$$(= \frac{1}{n^{2k+3}}\{\sum_{r=0}^{n-1}\int_0^1 P_{2k+2}(t)f^{(2k+2)}(\tfrac{t+r}{n})dt\}),$$

where the Bernoulli numbers B_n and the Bernoulli polynomials $P_n(t)$ are defined by the identities

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n \text{ and } \frac{x(e^{xt}-1)}{e^x-1} = \sum_{n=0}^{\infty} P_n(t)\,x^n.$$

1.8 A RESUME OF EXPONENTIAL-TYPE OPERATORS

In this section we mention some results from [2] and [45] on approximation in C-norm by exponential type operators and their duals. These will be useful both for comparison with L_p cases and in our later analysis. Also we derive some simple consequences of these.

A formula for the moments of exponential type operators ([45, Proposition 3.2], [2, Lemma 1.3.3]) is given by the following lemma.

Lemma 1.8.1. For $m \in \mathbb{N}$, $A_m(n,t)$ defined as

$$A_m(n,t) = n^m \int_A^B W(n,t,u)\,(u-t)^m\,du$$

is a polynomial in t of degree m, in n of degree $[\frac{m}{2}]$. The coefficient of n^m in $A_{2m}(n,t)$ is $(2m-1)!!\,p^m(t)$. And the coefficient of n^m in $A_{2m+1}(n,t)$ is $c_m\,p^m(t)\,p'(t)$, where $c_m = \frac{(2m+1)!!\,m}{3}$.

Corollary 1.8.2. Let r be a positive real number. Then, for all t belonging to a compact subset K of (A,B) there holds

$$|A_r(n,t)| \leq M\,n^{r/2}, \tag{1.8.1}$$

the constant M being independent of n.

The proof follows from Lemmas 1.7.5 and 1.8.1.

Next, we state a Lorentz-type lemma ([2, Lemma 1.5.6]) for the derivatives of the kernel $W(n,t,u)$ of an exponential type operator.

Lemma 1.8.3. For each $k \in \mathbb{N}$ there exist polynomials $a_{ij}^{(k)}(t)$ in t which do not depend on u or n such that

$$(1.8.2) \qquad \frac{\partial^k}{\partial t^k} W(n,t,u) = Q_k(n,t,u)\, W(n,t,u),$$

where $\quad Q_k(n,t,u) = \sum_{i,j} n^{i+j} (u-t)^j \, a_{ij}^{(k)}(t)\,(p(t))^{-k}$,

$i,j \in \mathbb{N}^0$ satisfy $2i+j \leq k$ and $p(t)$ is as defined in (1.5.6)

A function $\psi \in C(A,B)$ is called a growth test function for $\{S_n\}$ if for any compact subset K of $(A,B)'$ there exists a $n_o \in \mathbb{N}$ and a positive constant M such that

$$S_n(\psi^2,t) < M, \; n > n_o, \; t \in K.$$

Lemma 1.8.4. Let $|f(t)| \leq \psi(t)$, $t \in (A,B)$ for some GTF ψ. Then the relation $\lim_{n\to\infty} S_n(f,t) = f(t)$ holds at each continuity point of f. If f is continuous on $[a,b]$, then $\lim_{n\to\infty} S_n(f,t) = f(t)$ holds uniformly on compact subsets of (a,b).

An asymptotic formula for linear combinations $S_n(\cdot,k,t)$ of exponential type operators is given by May [45] :

Lemma 1.8.5. Let $|f(t)| \leq \psi(t)$, $t \in (A,B)$ for some growth test function ψ. If for some $t \in (A,B)$ $f^{(2k+2)}(t)$ exists, then

$$(1.8.3) \qquad n^{k+1}\{S_n(f,k,t)-f(t)\} = \sum_{j=k+1}^{2k+2} Q(j,k,t) f^{(j)}(t) + o(1),$$

where $Q(j,k,t)$ are certain polynomials in t. Moreover, $Q(2k+2,k,t) = c_1(p(t))^{k+1}$ and $Q(2k+1,k,t) = c_2(p(t))^k\, p'(t)$.

Furthermore, if $f \in C^{2k+2}[a,b]$, then (1.8.3) is uniform in every interval $[a_1,b_1] \subset (a,b)$.

Inverse and saturation theorems for linear combinations of exponential type operators in C-norm, over contracting intervals, have been proved by May [45]. We assume in the following two lemmas that $A < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < B$.

Lemma 1.8.6. Let $0 < \alpha < 2$ and $|f(t)| \leq \psi(t)$, $t \in (A,B)$ for some GTF ψ. Then, in the following, the implications ''(i) ==> (ii) ==> (iii)'' hold.

(i) $||S_n(f,k,t)-f(t)||_{C[a_1,b_1]} = O(n^{-\alpha(k+1)/2})$, $(n \to \infty)$;

(ii) $\omega_{2k+2}(f,\tau,[a_2,b_2]) = O(\tau^{\alpha(k+1)})$, $(\tau \to 0)$;

(iii) $||S_n(f,k,t)-f(t)||_{C[a_3,b_3]} = O(n^{-\alpha(k+1)/2})$, $(n \to \infty)$.

Lemma 1.8.7. If S_n are regular, then, in the following, the implications ''(i) ==> (ii) ==> (iii)'' and ''(iv) ==> (v) ==> (vi)'' hold.

(i) $||S_n(f,k,t)-f(t)||_{C[a_1,b_1]} = O(n^{-(k+1)})$, $(n \to \infty)$;

(ii) $f^{(2k+1)} \in A.C.[a_2,b_2]$ and $f^{(2k+2)} \in L_\infty[a_2,b_2]$;

(iii) $||S_n(f,k,t)-f(t)||_{C[a_3,b_3]} = O(n^{-(k+1)})$, $(n \to \infty)$;

(iv) $||S_n(f,k,t)-f(t)||_{C[a_1,b_1]} = o(n^{-(k+1)})$, $(n \to \infty)$;

(v) $f \in C^{2k+2}[a_2,b_2]$ and $\sum_{i=k+1}^{2k+2} Q(i,k,t)f^{(i)}(t) = 0$, where $t \in [a_2,b_2]$. $Q(i,k,t)$ are the polynomials occuring in (1.8.3);

(vi) $\|S_n(f,k,t)-f(t)\|_{C[a_3,b_3]} = o(n^{-(k+1)})$, $(n \to \infty)$.

The dual operators $S_n^*(\cdot,u)$ corresponding to exponential type operators $S_n(\cdot,t)$ have been defined and studied by Agrawal [2]. These are given by

(1.8.4) $$S_n^*(f,u) = \int_A^B W(n,t,u)\, f(t)\, dt.$$

The normalized kth moment $\sigma_{n,k}^*(u)$ is defined through

(1.8.5) $$\sigma_{n,k}^*(u) = (a(n))^{-1}\, S_n^*((t-u)^k,u).$$

A recursion relation for the normalized moments is given by Lemma 1.4.1 of [2, p. 24]:

Lemma 1.8.8. For each $k \in \mathbb{N}$ and all sufficiently large values of n there holds

(1.8.6) $$(n-(k+2)\alpha)\, \sigma_{n,k+1}^*(u) = 2(k+1)(\alpha u+\beta)\, \sigma_{n,k}^*(u) + kp(u)\, \sigma_{n,k-1}^*(u),$$

where $p(t) = \alpha t^2+\beta t+\gamma$ and $\sigma_{n,-1}^*(u) = 0$.

As a corollary one obtains the following estimate:

Corollary 1.8.9. For $k \in \mathbb{N}$ and all sufficiently large values of n, $S_n^*((t-u)^k,u)$ is a polynomial in u of degree $\leq k$. Moreover, for $u \in K$, a compact subset of (A,B), there holds for some constant M

(1.8.7) $$|S_n^*((t-u)^k,u)| \leq M\, n^{-\left[\frac{k+1}{2}\right]}.$$

As a consequence of the above corollary and Lemma 1.7.5 we obtain the following estimate:

Corollary 1.8.10. For any positive number r and $u \in K$, there holds for all sufficiently large values of n

$$(1.8.8) \qquad S_n^* (|t-u|^r, u) \leq M\, n^{-r/2},$$

M being a constant.

It has been proved in [2] that the operators $S_n^*(\cdot,u)$ are also approximation methods. Moreover, the following asymptotic relation for linear combinations of operators $S_n^*(\cdot,u)$ holds:

Lemma 1.8.11. Let $|f(u)| \leq \psi(u)$, $u \in (A,B)$ for some growth test function ψ. If for some $u \in (A,B)$ $f^{(2k+2)}(u)$ exists, then

$$(1.8.9) \qquad \lim_{n \to \infty} n^{k+1} \{ S_n^*(f,k,u) - f(u) \} = \sum_{j=0}^{2k+2} \frac{f^{(j)}(u)}{j!} Q^*(j,k,u),$$

where $Q^*(j,k,u)$ is the coefficient of $n^{-(k+1)}$ in the asymptotic expansion of $S_n^*((t-u)^j,u)$, multiplied by $(-1)^k / \prod_{i=0}^{k} d_i$. Moreover, (1.8.9) holds uniformly in $u \in [a,b]$ if $f^{(2k+2)}$ is continuous on $\langle a,b \rangle$.

May [45] proved and made use of the following lemma in the proof of sup-norm saturation theorem for $S_n(\cdot,k,t)$.

Lemma 1.8.12. For every $m \in \mathbb{N}^0$ there holds

$$(1.8.10) \qquad \int_A^B W(n,t,u)\, t^m dt = P(u,n) + O(n^{-k-2}),$$

where $P(u,n)$ is a polynomial in u and n^{-1}. The degree of $P(u,n)$ in u is m, and the O-term is uniform for $u \in [a,b] \subset (A,B)$.

In the proof of saturation theorem, for interpolatory modifications of regular exponential type operators in Chapter V, we require the coefficient of u^m in $P(u,n)$. This is obtained in the following lemma:

Lemma 1.8.13. Let $m \in \mathbb{N}$ and $a_m = a_m(n)$ be the coefficient of u^m in $P_m(u,n) = \int_A^B W(n,t,u)\, t^m\, dt$. Then

$$(1.8.11) \qquad a_m = a(n) \prod_{j=2}^{m+1} (1 - \frac{j}{n}\alpha)^{-1},$$

where α is the coefficient of t^2 in $p(t)$.

Proof. We proceed by an induction on m.

$$P_1(u,n) = \int_A^B W(n,t,u)t\, dt = \int_A^B W(n,t,u)(t-u)dt + u\int_A^B W(n,t,u)dt$$

$$= -\frac{1}{n} \int_A^B (\frac{\partial}{\partial t} W(n,t,u))\, p(t)dt + u\, a(n).$$

Let $p(t) = \alpha t^2 + \beta t + \gamma$, then by (1.5.10) and (1.5.8)

$$P_1(u,n) = -\frac{1}{n}\{-\int_A^B W(n,t,u)(2\alpha t+\beta)dt\} + u\, a(n)$$

$$= 2\frac{\alpha}{n} P_1(u,n) + \frac{\beta}{n} a(n) + u\, a(n)$$

i.e., $$P_1(u,n) = u\, a(n)(1- \frac{2\alpha}{n})^{-1} + \frac{\beta}{n} a(n)(1- \frac{2\alpha}{n})^{-1}.$$

This proves the result when $m=1$. We assume it true for $m=r$.

Then $P_{r+1}(u,n) = \int_A^B W(n,t,u)t^r(t-u)dt + \int_A^B W(n,t,u)ut^r dt$

$$= -\frac{1}{n}\{\int_A^B (\frac{\partial}{\partial t} W(n,t,u))t^r p(t)dt\} + uP_r(u,n).$$

Again integrating by parts and using (1.5.10)

$$P_{r+1}(u,n) = \frac{1}{n}\{\int_A^B W(n,t,u)(t^r p(t))' dt\} + uP_r(u,n).$$

Since $(t^r p(t))' = \alpha(r+2)t^{r+1} + \beta(r+1)t^r + \gamma r\, t^{r-1}$,

$$P_{r+1}(u,n) = \frac{\alpha(r+2)}{n} P_{r+1}(u,n) + uP_r(u,n)$$

$$+ \frac{\beta(r+1)}{n} P_r(u,n) + \gamma \frac{r}{n} P_{r-1}(u,n)$$

i.e., $P_{r+1}(u,n) = (1-\frac{\alpha(r+2)}{n})^{-1}\, uP_r(u,n)$

$$+(1-\frac{\alpha(r+2)}{n})^{-1} (\frac{\beta(r+1)}{n} P_r(u,n) + \gamma \frac{r}{n} P_{r-1}(u,n)).$$

By Lemma 1.8.12, $P_r(u,n)$ and $P_{r-1}(u,n)$ are polynomials in u of degrees r and r-1 respectively and hence

$$a_{r+1} = (1-\alpha \frac{(r+2)}{n})^{-1}\, a(n) \prod_{j=2}^{r+1} (1-\alpha \frac{j}{n})^{-1} = a(n) \prod_{j=2}^{r+2} (1-\alpha \frac{j}{n})^{-1},$$

completing proof of the lemma.

The above mentioned results concern ordinary approximation by exponential type operators. An interesting fact about these operators is that they also have the simultaneous approximation property $S_n^{(k)}(f,t) \to f^{(k)}(t)$, $t \in (A,B)$, $k \in \mathbb{N}$. Inverse and saturation theorems in simultaneous approximation for some sequences of linear positive operators have been proved in [2], [37], [57], [58] and [65].

CHAPTER II

L_p-APPROXIMATION BY LINEAR COMBINATIONS OF BERNSTEIN-KANTOROVITCH POLYNOMIALS

A study of direct, inverse and saturation theorems for Bernstein-Kantorovitch polynomials in L_p-norm $(1 \leq p < \infty)$ has been made in [25] (see 1.7). Here we obtain these theorems for the linear combinations of Bernstein-Kantorovitch polynomials in L_p-norm $(1 \leq p < \infty)$. Our results are local in nature over contracting intervals. The proofs of some of these results require estimates of adjoint moments of Bernstein-Kantorovitch polynomials which we prove in Section 1. In Section 2 we show that the linear conbinations of Bernstein-Kantorovitch polynomials in L_p-norm converge faster to function provided function is sufficiently smooth. In Section 3 we prove an inverse theorem related to $P_n(\cdot,k,t)$. The proof is carried out by using properties of a linear method of approximation (viz., Steklov means). In Section 4 we prove a saturation theorem.

We use the notations $I = [0,1]$, $I_j = [a_j,b_j]$, $j=1,2,3$, where $0 < a_j < a_{j+1}$ and $b_{j+1} < b_j < 1$, throughout this and the next chapter.

2.1 The Dual Operators

Let $\{P_n\}$ be the sequence of Bernstein-Kantorovitch polynomial operators. We define the dual operator sequence $\{P_n^*\}$ as follows (the kernel $K(n,t,u)$ is defined in Section 1.5)

$$(2.1.1) \qquad P_n^*(f,u) = \int_0^1 K(n,t,u)\, f(t)\, dt.$$

The domain of the operators P_n^* is the set $L_1(I)$. Let $\langle f,g\rangle$ denote the real inner product $\int_0^1 f(u)\, g(u)\, du$. Then

$$\begin{aligned}
\langle P_n^*(f,u), g(u)\rangle &= \int_0^1 P_n^*(f,u) g(u) du \\
&= \int_0^1 \int_0^1 K(n,t,u)\, f(t) g(u) dt\, du \\
&= \int_0^1 \int_0^1 K(u,t,u) f(t) g(u)\, du\, dt, \text{by Fubini's theorem} \\
&= \int_0^1 P_n(g,t) f(t) dt = \langle P_n(g,t), f(t)\rangle.
\end{aligned}$$

Note that for each $f \in L_1(I)$, $P_n^*(f,u)$ is a step function. Moreover, for $f \in L_p(I)$, $1 \leq p < \infty$, making use of the Jensen's inequality and then Fubini's theorem we have

$$\begin{aligned}
||P_n^*(f,u)||^p_{L_p(I)} &= \int_0^1 \left|\int_0^1 K(n,t,u) f(t) dt\right|^p du \\
&\leq \int_0^1 \int_0^1 K(n,t,u)\, |f(t)|^p\, dt\, du \\
&= \int_0^1 \int_0^1 K(n,t,u) |f(t)|^p\, du\, dt \\
&= \int_0^1 |f(t)|^p \left(\int_0^1 K(n,t,u) du\right) dt
\end{aligned}$$

$$= ||f||^p_{L_p(I)}.$$

Hence we have the inequality

(2.1.2) $||P_n^*(f,u)||_{L_p(I)} \leq ||f||_{L_p(I)}$, $f \in L_p(I)$.

By definition

$$P_n^*(1,u) = 1, \ u \in I,$$

and $$P_n^*((t-u)^2,u) = \sum_{\nu=0}^{n} \chi_{n\nu}(u)\left\{\frac{(\nu+1)(\nu+2)}{(n+2)(n+3)} - \frac{2u(\nu+1)}{(n+2)} + u^2\right\}, u \in I$$

$$= O(\tfrac{1}{n}), \quad (n \to \infty),$$

uniformly in $u \in I$.

Hence it follows from Theorem 3 of Korovkin [36] that $P_n^*(f,u) \to f(u)$, uniformly in $u \in I$ as $n \to \infty$, provided $f \in C(I)$. Therefore, by (2.1.2) the operators P_n^* constitute an approximation method also for $f \in L_p(I)$,

i.e., $||P_n^*(f,u) - f(u)||_{L_p(I)} = o(1)$, $(n \to \infty)$.

Let $k \in \mathbb{N}$. Then the k-th dual moment is given by $\mu^*_{n,k}(u) = P_n^*((t-u)^k,u)$. Their bounds can be obtained from

Proposition 2.1.1. Let $k \in \mathbb{N}$ and $u \in I_1$. Then

(2.1.3) $P_n^*(|t-u|^k,u) = O(n^{-k/2})$, $(n \to \infty)$,

uniformly in $u \in I_1$.

Proof. It is sufficient to show that for $r \in \mathbb{N}^0$, $[c,d] \subset (0,1)$ and for all $\nu(=1,\dots,n-1)$ satisfying $c < \frac{\nu}{n} < d$, there holds for some constant M independent of n and ν

$$(2.1.4) \qquad \int_0^1 p_{n\nu}(t)\left|\frac{\nu}{n}-t\right|^r dt \le \frac{M}{n^{\frac{r}{2}+1}} .$$

Now, let $u \in I_1$. Then $u \in [\frac{\nu'}{n+1}, \frac{\nu'+1}{n+1})$ for some $\nu' \in \{0,1,\dots,n\}$. Thus

$$P_n^*(|t-u|^k,u) = (n+1)\{\sum_{\nu=0}^{n} \chi_{n\nu}(u)(\int_0^1 p_{n\nu}(t)|t-u|^k\,dt)\}$$

$$= (n+1)\int_0^1 p_{n\nu'}(t)|t-u|^k dt \ (\text{as } u \in [\tfrac{\nu'}{n+1},\tfrac{\nu'+1}{n+1}))$$

$$= (n+1)\int_0^1 p_{n\nu'}(t)\left|t-\frac{\nu'}{n}+\frac{\nu'}{n}-u\right|^k dt$$

$$\le (n+1)\int_0^1 p_{n\nu'}(t)\left(\left|t-\frac{\nu'}{n}\right|+\left|\frac{\nu'}{n}-u\right|\right)^k dt$$

$$(2.1.5) \qquad = (n+1)\{\sum_{j=0}^{k}\binom{k}{j}\int_0^1 p_{n\nu'}(t)\left|\frac{\nu'}{n}-t\right|^j\left|\frac{\nu'}{n}-u\right|^{k-j}dt\}.$$

We note that

$u \in [\frac{\nu'}{n+1}, \frac{\nu'+1}{n+1})$ implies that $|u-\frac{\nu'}{n}| \le \frac{1}{n}$ and hence

if we assume that (2.1.4) holds, then by (2.1.5)

$$P_n^*(|t-u|^k,u) \le (n+1)\{\sum_{j=0}^{k}\binom{k}{j}\frac{1}{n^{k-j}}\cdot\frac{M}{n^{\frac{j}{2}+1}}\} \le \frac{M'}{n^{k/2}} .$$

Proof of (2.1.4). Consider those ν's which satisfy

$$(2.1.6) \qquad c < \frac{\nu}{n} < d.$$

For $\theta \in (0,1)$ we define a function $\beta(t)$ as follows:

$$\beta(t) = \begin{cases} \dfrac{(t+\theta)^{\theta}\,(1-\theta-t)^{1-\theta}}{\theta^{\theta}(1-\theta)^{1-\theta}} & , t \in [-\theta, 1-\theta], \\ 0 & , t \notin [-\theta, 1-\theta]. \end{cases}$$

Then $\beta(t)$ is a bell-shaped function (see Section 1.4). Also it is easily seen that $\beta^{(2)}(0) = -(\theta(1-\theta))^{-1}$. An application of Lemma 1.4.1 to the function $\beta(t)$ gives

$$\lim_{n \to \infty} n^{(r+1)/2} \int_{-\theta}^{1-\theta} |t|^r \frac{(t+\theta)^{n\theta}\,(1-t-\theta)^{n(1-\theta)}}{\theta^{n\theta}\,(1-\theta)^{n(1-\theta)}}\, dt$$

$$= \Gamma\left(\frac{r+1}{2}\right)\left(\frac{-2}{\beta^{(2)}(0)}\right)^{(r+1)/2} = M_1 \text{ (say)}.$$

By a transformation of the variable we obtain

$$(2.1.7) \qquad \lim_{n \to \infty} n^{(r+1)/2} \int_0^1 |t-\theta|^r \frac{t^{n\theta}(1-t)^{n(1-\theta)}}{\theta^{n\theta}(1-\theta)^{n(1-\theta)}}\, dt = M_2.$$

By Stirling's formula for large positive values of x

$$\Gamma(x+1) \cong (2\pi x)^{1/2}\, x^x\, e^{-x}.$$

This implies that

$$(2.1.8) \qquad \frac{\Gamma(n+1)}{\Gamma(n\theta+1)\,\Gamma(n-n\theta+1)} \cong \frac{\{2\pi n\theta(1-\theta)\}^{-1/2}}{\{\theta^{\theta}(1-\theta)^{1-\theta}\}^{n}}.$$

(2.1.7) and (2.1.8) imply that as n tends to infinity

$$(2.1.9) \qquad \frac{\Gamma(n+1)}{\Gamma(n\theta+1)\,\Gamma(n-n\theta+1)} \int_0^1 |t-\theta|^r\, t^{n\theta}(1-t)^{n(1-\theta)} dt \leq \frac{M_2}{n^{\frac{r}{2}+1}}.$$

Choosing $\theta = \frac{\nu}{n}$ where $\frac{\nu}{n}$ satisfies (2.1.6) we obtain from (2.1.9)

$$\binom{n}{\nu} \int_0^1 t^\nu (1-t)^{n-\nu} \left|\frac{\nu}{n} - t\right|^r dt \leq \frac{M_2}{n^{\frac{r}{2}+1}} .$$

This proves (2.1.4).

Corollary 2.1.2. Let $r > 0$. Then for all ν's satisfying (2.1.6) there holds

$$(2.1.10) \qquad \int_0^1 p_{n\nu}(t) \left|\frac{\nu}{n} - t\right|^r dt \leq \frac{M}{n^{\frac{r}{2}+1}} ,$$

where M is a constant independent of n and ν.

Proof. Let $s > r$ be an integer. Using Holder's inequality we obtain from (2.1.4)

$$\int_0^1 p_{n\nu}(t)\left|\frac{\nu}{n} - t\right|^r dt \leq \left\{\int_0^1 p_{n\nu}(t)\left|\frac{\nu}{n} - t\right|^s dt\right\}^{r/s} \left\{\int_0^1 p_n(t)dt\right\}^{1-r/s}$$

$$\leq \left(\frac{M_1}{n^{\frac{s}{2}+1}}\right)^{r/s} \frac{1}{n^{1-r/s}}$$

$$= \frac{M}{n^{\frac{r}{2}+1}} .$$

2.2 ERROR ESTIMATES AND A DIRECT THEOREM

In this section first we obtain estimates for error in L_p-approximation $(1 \leq p < \infty)$ by linear combinations $P_n(.,k,t)$ of Bernstein-Kantorovitch polynomials in terms of L_p-norm of derivatives of the function. The proof of the case $p > 1$ makes use of Lemma 1.2.1 regarding Hardy-Littlewood majorant of the function and Lemma 1.2.2 which bounds intermediate derivatives of the function in terms of a higher derivative and the function in L_p-norm. While, when $p = 1$, we use Lemma 1.2.5 regarding integration by parts in Lebesgue-Stieltjes integration and Lemma 1.2.2. Using these results we obtain a general error estimate in terms of (2k+2)th integral modulus of smoothness of the function. Finally we prove Voronovskaja type asymptotic formula for the operators $P_n(.,k,t)$.

Theorem 2.2.1. Let $1 < p < \infty$ and $f \in L_p(I)$. If f has 2k+2 derivatives on I_1 with $f^{(2k+1)} \in A.C.(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$, then

$$(2.2.1)\quad ||P_n(f,k,t)-f(t)||_{L_p(I_2)} \leq \frac{M}{n^{k+1}}\{||f^{(2k+2)}||_{L_p(I_1)} + ||f||_{L_p(I)}\},$$

where M is a certain constant.

To prove the theorem we require the following proposition

Proposition 2.2.2. Let $1 < p < \infty$, $h \in L_p(I)$ and $i,j \in \mathbb{N}^o$. Then for a sufficiently large positive number ℓ

$$(2.2.2) \qquad \|P_n(|u-t|^i|\int_t^u |u-w|^j|h(w)|dw|, t)\|_{L_p(I_2)}$$

$$\leq M\{n^{-(\frac{i+j+1}{2})}\|h\|_{L_p(I_1)} + n^{-q}\|h\|_{L_p(I)}\},$$

where M is a certain constant.

Furthermore, $\|P_n(|u-t|^i|\int_t^u |u-w|^j|h(w)|dw|, t)\|_{L_p(I)}$

$$(2.2.3) \qquad \leq M_1\, n^{-(\frac{i+j+1}{2})}\|h\|_{L_p(I)},$$

where M_1 is another constant.

Proof. Let $\chi(u)$ denote the characteristic function of I_1. Then

$$\|P_n(|u-t|^i|\int_t^u |u-w|^j|h(w)|dw|, t)\|_{L_p(I_2)}$$

$$\leq \|P_n(\chi(u)|u-t|^i|\int_t^u |u-w|^j|h(w)|dw|, t)\|_{L_p(I_2)}$$

$$+\|P_n((1-\chi(u))|u-t|^i|\int_t^u |u-w|^j|h(w)|dw|, t)\|_{L_p(I_2)}$$

$$= J_1 + J_2, \text{ say.}$$

First we evaluate J_1. Let $H_h(u)$ be the Hardy-Littlewood majorant of h defined by (1.2.1). Consider the following inequality:

$$\int_{a_1}^{b_1} K(n,t,u)|u-t|^i|\int_t^u |u-w|^j|h(w)|dw|du$$

$$\leq \int_{a_1}^{b_1} K(n,t,u)|u-t|^{i+j+1}(\frac{1}{|u-t|}|\int_t^u |h(w)|dw|)du$$

$$\leq \int_{a_1}^{b_1} K(n,t,u)|u-t|^{i+j+1} H_h(u)\, du.$$

Now using Holder's inequality (with $p^{-1}+q^{-1} = 1$) and then applying Corollary 1.7.7 in the next step we get

$$\int_{a_1}^{b_1} K(n,t,u)|u-t|^{i+j+1} H_h(u)du$$

$$\leq \{\int_{a_1}^{b_1} K(n,t,u)|u-t|^{(i+j+1)q} du\}^{1/q}\{\int_{a_1}^{b_1} K(n,t,u)|H_h(u)|^p du\}^{1/p}$$

$$\leq M_2\, n^{-(\frac{i+j+1}{2})} \{\int_{a_1}^{b_1} K(n,t,u)|H_h(u)|^p du\}^{1/p}.$$

Using Fubini's theorem and Lemma 1.2.1 it follows that

$$J_1 \leq M_2\, n^{-(\frac{i+j+1}{2})} \{\int_{a_2}^{b_2}\int_{a_1}^{b_1} K(n,t,u)|H_h(u)|^p du\, dt\}^{1/p}$$

$$= M_2\, n^{-(\frac{i+j+1}{2})} \{\int_{a_1}^{b_1} |H_h(u)|^p (\int_{a_2}^{b_2} K(n,t,u)dt)du\}^{1/p}$$

$$\leq M_2'\, n^{-(\frac{i+j+1}{2})} ||h||_{L_p(I_1)}.$$

Next we estimate J_2. Let $\delta = \min(a_2-a_1, b_1-b_2)$. Using Jensen's inequality twice and writing $s = 2\ell p + 1 - (i+j+1)p$ we have

$$J_2^p \leq \int_{a_2}^{b_2}\int_{0}^{1} K(n,t,u)|u-t|^{ip}(1-\chi(u))^p|\int_t^u |u-w|^j|h(w)|dw|^p du\, dt$$

$$\leq \int_{a_2}^{b_2}\int_{0}^{1} K(n,t,u)|u-t|^{(i+1)p-1}(1-\chi(u))^p|\int_t^u |u-w|^{jp}|h(w)|^p dw|du dt$$

$$\leq \int_{a_2}^{b_2}\int_{0}^{1} K(n,t,u)|u-t|^{(i+j+1)p-1}(1-\chi(u))^p|\int_t^u |h(w)|^p dw|du\, dt$$

$$\leq \delta^{-s}\int_{a_2}^{b_2}\int_{0}^{1} K(n,t,u)|u-t|^{(i+j+1)p+s-1}(1-\chi(u))^p|\int_t^u |h(w)|^p dw|du dt$$

$$\leq \delta^{-s} \, ||h||^p_{L_p(I)} \int_{a_2}^{b_2} \int_0^1 K(n,t,u) \, |u-t|^{2\ell p} \, du \, dt.$$

Making use of Corollary 1.7.7 it follows that

$$J_2 \leq \frac{M_3}{n^\ell} \, ||h||_{L_p(I)}.$$

Finally, (2.2.2) follows from estimates of J_1 and J_2. The proof of (2.2.3) follows along the lines of the above proof but for one change that we do not have to decompose integration in u into the two parts denoted J_1 and J_2.

Proof of Theorem 2.2.1. For $t \in I_2$ and $u \in I_1$, with the given assumptions on f we can write ([30, Corollary 18.20])

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w) \, dw.$$

Hence, if $\chi(u)$ is the characteristic function of I_1, we have

$$P_n(f,t) = \int_0^1 K(n,t,u) \, f(u) \, du$$

$$= \int_0^1 K(n,t,u)\chi(u)f(u)du + \int_0^1 K(n,t,u)(1-\chi(u))f(u)du$$

$$= \sum_{i=0}^{2k+1} \left\{\frac{f^{(i)}(t)}{i!} \int_0^1 K(n,t,u) \, \chi(u) \, (u-t)^i \, du\right\}$$

$$+ \frac{1}{(2k+1)!} \int_0^1 K(n,t,u)\chi(u) \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w) \, dw \, du$$

$$+ \int_0^1 K(n,t,u)(1-\chi(u)) \, f(u) \, du$$

$$= \sum_{i=0}^{2k+1} \frac{f^{(i)}(t)}{i!} \int_0^1 K(n,t,u)(u-t)^i \, du$$

$$+\frac{1}{(2k+1)!}\int_0^1 K(n,t,u)\chi(u)\int_t^u (u-w)^{2k+1}f^{(2k+2)}(w)dw\,du$$

$$+\int_0^1 K(n,t,u)(1-\chi(u))\,f(u)\,du$$

$$+\sum_{i=0}^{2k+1}\{\frac{f^{(i)}(t)}{i!}\int_0^1 K(n,t,u)(\chi(u)-1)(u-t)^i\,du\}$$

Therefore,

$$P_n(f,k,t)-f(t)=\{\sum_{i=1}^{2k+1}\frac{f^{(i)}(t)}{i!}\{\sum_{j=0}^{k}c(j,k)\int_0^1 K(d_jn,t,u)(u-t)^i\,du\}\}$$

$$+\frac{1}{(2k+1)!}\{\sum_{j=0}^{k}c(j,k)\{\int_0^1 K(d_jn,t,u)\chi(u)\times$$

$$\times\{\int_t^u (u-w)^{2k+1}f^{(2k+2)}(w)dw\,du\}\}\}$$

$$+\{\sum_{j=0}^{k}c(j,k)\{\int_0^1 K(d_jn,t,u)(1-\chi(u))\,f(u)\,du\}\}$$

$$+\{\sum_{i=0}^{2k+1}\frac{f^{(i)}(t)}{i!}\{\sum_{j=0}^{k}c(j,k)\{\int_0^1 K(d_jn,t,u)(\chi(u)-1)(u-t)^i du\}\}\}$$

$$(2.2.4)\quad =\Sigma_1(t)+\Sigma_2(t)+\Sigma_3(t)+\Sigma_4(t),\ \text{say.}$$

In view of the fact that $\sum_{j=0}^{k} c(j,k)d_j^{-m}=0, m=1,2,\ldots,k$, we obtain from Corollary 1.7.3, $||\Sigma_1||_{L_p(I_2)}\le\frac{M_1}{n^{k+1}}(\sum_{i=1}^{2k+1}||f^{(i)}||_{L_p(I_2)})$.

Making use of Lemma 1.2.2 we obtain another bound for the expression on the right side, i.e.,

$$||\Sigma_1||_{L_p(I_2)}\le\frac{M_1'}{n^{k+1}}(||f^{(2k+2)}||_{L_p(I_2)}+||f||_{L_p(I_2)}),$$

where M_1 and M_1' are certain constants.

The estimate $||\Sigma_2||_{L_p(I_2)} \leq \frac{M_2}{n^{k+1}} ||f^{(2k+2)}||_{L_p(I_1)}$

follows from the estimate of J_1 in Proposition 2.2.2.
Taking $\ell = k+1$ in Lemma 1.7.9 and using boundedness of $c(j,k)$'s we have

$$||\Sigma_3||_{L_p(I_2)} \leq \frac{M_3}{n^{k+1}} ||f||_{L_p(I)}.$$

Due to the presence of the factor $(\chi(u)-1)$ it easily follows from Lemma 1.7.9 that for all $t \in I_2$

$$\int_0^1 K(n,t,u)\ (\chi(u)-1)(u-t)^i\ du \leq \frac{M_4}{n^{k+1}} .$$

This,alongwith Lemma 1.2.2,implies that

$$||\Sigma_4||_{L_p(I_2)} \leq \frac{M_4'}{n^{k+1}} (||f||_{L_p(I_2)} + ||f^{(2k+2)}||_{L_p(I_2)}).$$

The theorem now follows from (2.2.4) and the estimates of Σ_1, Σ_2, Σ_3 and Σ_4.

Corollary 2.2.3. Let $1 < p < \infty$ and $f \in L_p(I)$. If f has $2k+2$ derivatives on I with $f^{(2k+1)} \in A.C.(I)$ and $f^{(2k+2)} \in L_p(I)$, then for some constant M

$$(2.2.5)\quad ||P_n(f,k,t)-f(t)||_{L_p(I)} \leq \frac{M}{n^{k+1}} \{||f^{(2k+2)}||_{L_p(I)} + ||f||_{L_p(I)}\}.$$

Proceeding as in the proof of Theorem 2.2.1 and using the second assertion (2.2.3) of Proposition 2.2.2 we obtain its proof.

Theorem 2.2.4. Let $f \in L_1(I)$. If f has $2k+1$ derivatives on I_1 with $f^{(2k)} \in$ A.C. (I_1) and $f^{(2k+1)} \in$ B.V.(I_1), then for some constant M

$$(2.2.6)\quad ||P_n(f,k,t)-f(t)||_{L_1(I_2)} \le \frac{M}{n^{k+1}}\{||f^{(2k+1)}||_{B.V.(I_1)}+||f^{(2k+1)}||_{L_1(I_2)} + ||f||_{L_1(I)}\}.$$

First we prove an auxiliary result from which the proof of the theorem will follow easily.

Proposition 2.2.5. Let $h \in$ B.V.(I). Then for $i, j \in \mathbb{N}^o$ and a fixed positive number ℓ

$$(2.2.7)\quad ||P_n(|u-t|^i|\int_t^u |u-w|^j|dh(w)||,t)||_{L_1(I_2)} \le M_1\{n^{-(i+j+1)/2}||h||_{B.V.(I_1)}+n^{-\ell}||h||_{B.V.(I)}\},$$

where M_1 is a constant.

Furthermore, $||P_n(|u-t|^i|\int_t^u |u-w|^j|dh(w)||,t)||_{L_1(I)}$

$$\le M_2\, n^{-(i+j+1)/2}||h||_{B.V.(I)},$$

where M_2 is another constant.

Proof. Let $\chi(u)$ be characteristic function of I_1. Then

$$||P_n(|u-t|^i|\int_t^u |u-w|^j|dh(w)||,t)||_{L_1(I_2)}$$

$$= \int_{a_2}^{b_2} \int_0^1 K(n,t,u)|u-t|^i \left|\int_t^u |u-w|^j |dh(w)|\right| du\, dt$$

$$\leq \int_{a_2}^{b_2} \int_0^1 K(n,t,u)|u-t|^{i+j} \left|\int_t^u |dh(w)|\right| du\, dt$$

$$= \int_{a_2}^{b_2} \int_0^1 \chi(u)K(n,t,u)|u-t|^{i+j} \left|\int_t^u |dh(w)|\right| du\, dt$$

$$+ \int_{a_2}^{b_2} \int_0^1 (1-\chi(u))K(n,t,u)|u-t|^{i+j} \left|\int_t^u |dh(w)|\right| du\, dt$$

$$= J_1 + J_2, \text{ say}$$

First we estimate J_2. We note that the variables u and t are such that $|u-t| \geq \delta$ where $\delta = \min(b_1-b_2, a_2-a_1)$. Hence choosing ℓ sufficiently large so that $s = 2\ell - (i+j) > 0$ we have

$$J_2 \leq \int_{a_2}^{b_2} \int_0^1 (1-\chi(u))K(n,t,u) \frac{|u-t|^{2\ell}}{\delta^s} \left|\int_t^u |dh(w)|\right| du\, dt$$

$$\leq M_3 \, ||h||_{B.V.(I)} \left(\int_0^1 K(n,t,u)|u-t|^{2\ell} \, du\right).$$

Applying Corollary 1.7.7 we obtain

$$J_2 \leq M_3' \, n^{-\ell} \, ||h||_{B.V.(I)}.$$

For each n there exists a nonnegative integer $r = r(n)$ such that $\frac{r}{n^{1/2}} \leq \max\{b_1-a_2, b_2-a_1\} \leq \frac{r+1}{n^{1/2}}$.

We have

$$J_1 = \int_{a_2}^{b_2} \int_{a_1}^{b_1} K(n,t,u)|u-t|^{i+j} \left|\int_t^u |dh(w)|\right| du\, dt$$

$$= \int_{a_2}^{b_2} \int_{a_1}^{b_1} K(n,t,u)|u-t|^{i+j} \left|\int_t^u \chi(w)|dh(w)|\right| \, du\, dt$$

$$\leq \sum_{\ell=0}^{r} \int_{a_2}^{b_2} \{ \int_{t+\ell n^{-1/2}}^{t+(\ell+1)n^{-1/2}} K(n,t,u)|u-t|^{i+j} (\int_t^{t+(\ell+1)n^{-1/2}} \chi(w)|dh(w)|)\, du$$

$$+ \int_{t-(\ell+1)n^{-1/2}}^{t-\ell n^{-1/2}} K(n,t,u)|u-t|^{i+j} (\int_{t-(\ell+1)n^{-1/2}}^{t} \chi(w)|dh(w)|) du \}\, dt.$$

Let $\chi_{t,c,d}(w)$ denote characteristic function of interval $[t - cn^{-1/2}, t + dn^{-1/2}]$ where $c,d \in \mathbb{N}^0$. Then

$$J_1 \leq \sum_{\ell=1}^{r} \int_{a_2}^{b_2} \{ \int_{t+\ell n^{-1/2}}^{t+(\ell+1)n^{-1/2}} K(n,t,u)\, n^2 \ell^{-4} |u-t|^{i+j+4} \times$$

$$\times \{ \int_t^{t+(\ell+1)n^{-1/2}} \chi(w)\chi_{t,0,\ell+1}(w)|dh(w)| \}\, du$$

$$+ \int_{t-(\ell+1)n^{-1/2}}^{t-\ell n^{-1/2}} K(n,t,u)n^2 \ell^{-4}|u-t|^{i+j+4} \times$$

$$\times \{ \int_{t-(\ell+1)n^{-1/2}}^{t} \chi(w)\chi_{t,0,\ell+1}(w)|dh(w)| \}\, du \} dt$$

$$+ \int_{a_2}^{b_2} \int_{a_2-n^{-1/2}}^{b_2+n^{-1/2}} K(n,t,u)|u-t|^{i+j} \times$$

$$\times \{ \int_{t-n^{-1/2}}^{t+n^{-1/2}} \chi(w)\chi_{t,1,1}(w)|dh(w)| \} du\, dt$$

$$\leq \sum_{\ell=1}^{r} \{ \frac{n^2}{\ell^4} \int_{a_2}^{b_2} \{ \int_{t+\ell n^{-1/2}}^{t+(\ell+1)n^{-1/2}} K(n,t,u)|u-t|^{i+j+4} \times$$

$$\times (\int_{a_1}^{b_1} \chi_{t,0,\ell+1}(w)|dh(w)|)\, du +$$

$$+\int_{t-(\ell+1)n^{-1/2}}^{t-\ell n^{-1/2}} K(n,t,u)|u-t|^{i+j+4} \int_{a_1}^{b_1} \chi_{t,\ell+1,0}(w)|dh(w)|du\} dt\}$$

$$(2.2.8) \qquad + \int_{a_2}^{b_2} \int_{a_2-n^{-1/2}}^{b_2+n^{-1/2}} K(n,t,u)|u-t|^{i+j}\Big(\int_{a_1}^{b_1} \chi_{t,1,1}(w)|dh(w)|\Big)du\,dt.$$

We use moment estimates given by Corollary 1.7.7 to obtain a bound for $\int_0^1 K(n,t,u)|u-t|^{i+j+4}\,du$ and in the next step we apply Fubini's theorem. Thus

$$J_1 \le M_4 n^{-(i+j)/2}\{\sum_{\ell=1}^{r} \frac{1}{\ell^4}\{\int_{a_2}^{b_2}\int_{a_1}^{b_1} \chi_{t,0,\ell+1}(w)|dh(w)|\,dt$$

$$+\int_{a_2}^{b_2}\int_{a_1}^{b_1} \chi_{t,\ell+1,0}(w)|dh(w)|dt\} + \int_{a_2}^{b_2}\int_{a_1}^{b_1} \chi_{t,1,1}(w)|dh(w)|dt\}$$

$$= M_4\, n^{-(i+j)/2}\{\sum_{\ell=1}^{r}\frac{1}{\ell^4}\{\int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} \chi_{t,0,\ell+1}(w)dt\Big)|dh(w)|$$

$$+\int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} \chi_{t,\ell+1,0}(w)dt\Big)|dh(w)|\} + \int_{a_1}^{b_1}\Big(\int_{a_2}^{b_2} \chi_{t,1,1}(w)dt\Big)|dh(w)|\}$$

$$= M_4\, n^{-(i+j)/2}\{\sum_{\ell=1}^{r}\frac{1}{\ell^4}\{\int_{a_1}^{b_1}|dh(w)|\Big(\int_{w-(\ell+1)n^{-1/2}}^{w} dt\Big)$$

$$+\int_{a_1}^{b_1}|dh(w)|\Big(\int_{w}^{w+(\ell+1)n^{-1/2}} dt\Big)\} + \int_{a_1}^{b_1}|dh(w)|\Big(\int_{w-n^{-1/2}}^{w+n^{-1/2}} dt\Big)\}$$

$$\le M_4\, n^{-(i+j+1)/2}\Big(4\Big(\sum_{\ell=1}^{r}\frac{1}{\ell^3}\Big)+2\Big)\|h\|_{B.V.(I_1)}$$

$$\le M_4'\, n^{-(i+j+1)/2}\,\|h\|_{B.V.(I_1)}.$$

The estimates of J_1 and J_2 complete the proof of (2.2.7).

The second assertion of Proposition 2.2.5 follows in a similar fashion.

Proof of Theorem 2.2.4. Since $f^{(2k+1)} \in B.V.(I_1)$, it follows from Theorem 17.17 of [30] that $f^{(2k+1)}$ is continuous a.e. on I_1. This alongwith Theorem 14.1 of [61] implies that for almost all values of $t \in I_2$ and all values of $u \in I_1$

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} df^{(2k+1)}(w).$$

Let $\chi(u)$ be the characteristic function of I_1. Then writing $f = f\chi + f(1-\chi)$ and using the above equation in the next step, we have for almost all values of $t \in I_2$,

$$\begin{aligned}
P_n(f,t) &= P_n(\chi f,t) + P_n((1-\chi)f,t) \\
&= \sum_{i=0}^{2k+1} \{\frac{f^{(i)}(t)}{i!} P_n(\chi(u)(u-t)^i,t)\} + P_n((1-\chi) f,t) \\
&\quad + \frac{1}{(2k+1)!} P_n(\chi(u) \int_t^u (u-w)^{2k+1} df^{(2k+1)}(w),t) \\
&= \sum_{i=0}^{2k+1} \{\frac{f^{(i)}(t)}{i!} P_n((u-t)^i,t)\} + P_n((1-\chi)f,t) \\
&\quad + \frac{1}{(2k+1)!} P_n(\chi(u) \int_t^u (u-w)^{2k+1} df^{(2k+1)}(w),t) \\
&\quad + \sum_{i=0}^{2k+1} \{\frac{f^{(i)}(t)}{i!} P_n((\chi(u)-1)(u-t)^i,t)\}
\end{aligned}$$

Since in the Lebesgue integral deleting a set of measure zero is immaterial, we obtain from above

$$\|P_n(f,k,t)-f(t)\|_{L_1(I_2)}$$

$$\leq \sum_{i=1}^{2k+1} \left\{ \frac{1}{i!} \|f^{(i)}(t)\{P_n((u-t)^i,k,t)\}\|_{L_1(I_2)} \right\}$$

$$+ \|P_n((1-\chi)f,k,t)\|_{L_1(I_2)}$$

$$+ \frac{1}{(2k+1)!} \|P_n(\chi(u)\int_t^u (u-w)^{2k+1} df^{(2k+1)}(w),k,t)\|_{L_1(I_2)}$$

$$+ \sum_{i=0}^{2k+1} \left\{ \frac{1}{i!} \| f^{(i)}(t)\{P_n((\chi(u)-1)(u-t)^i,k,t)\}\|_{L_1(I_2)} \right\}$$

(2.2.9) $\qquad = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$, say.

Proceeding as in the proof of estimates of Σ_1, Σ_3 and Σ_4 of Theorem 2.2.1, we obtain

$$\Sigma_1 \leq \frac{M_5}{n^{k+1}} (\|f^{(2k+1)}\|_{L_1(I_2)} + \|f\|_{L_1(I_2)}),$$

$$\Sigma_2 \leq \frac{M_5'}{n^{k+1}} \|f\|_{L_1(I)},$$

and

$$\Sigma_4 \leq \frac{M_6}{n^{k+1}} (\|f^{(2k+1)}\|_{L_1(I_2)} + \|f\|_{L_1(I_2)}).$$

Finally, using the first assertion of Proposition 2.2.5

$$\Sigma_3 \leq \frac{M_6'}{n^{k+1}} \|f^{(2k+1)}\|_{B.V.(I_1)}.$$

The proof follows from (2.2.9) and the estimates of Σ_1 to Σ_4.

Proceeding in the manner of the proof of above Theorem 2.2.4 and using second assertion of Proposition 2.2.5 we get

Corollary 2.2.6. Let $f \in L_1(I)$. If f has derivatives to the order $2k+1$, $f^{(2k)} \in A.C.(I)$ and $f^{(2k+1)} \in B.V.(I)$, then there holds for some constant M

$$(2.2.10) \quad ||P_n(f,k,t)-f(t)||_{L_1(I)} \leq \frac{M}{n^{k+1}} \{ ||f^{(2k+1)}||_{B.V.(I)} + ||f^{(2k+1)}||_{L_1(I)} + ||f||_{L_1(I)} \}.$$

Theorem 2.2.7. Let $1 \leq p < \infty$ and $f \in L_p(I)$. Then for sufficiently large values of n

$$(2.2.11) \quad ||P_n(f,k,t) - f(t)||_{L_p(I_2)} \leq M \{ \omega_{2k+2}(f,n^{-1/2},p,I_1) + n^{-(k+1)} ||f||_{L_p(I)} \},$$

where M is a constant.

Proof. Let $f_{\eta,2k+2}(u)$ be the Steklov mean of $(2k+2)$th order corresponding to $f(u)$ where $\eta > 0$ is sufficiently small and $f(u)$ is defined as zero outside I. Then

$$||P_n(f,k,t)-f(t)||_{L_p(I_2)} \leq ||P_n(f-f_{\eta,2k+2},k,t)||_{L_p(I_2)} + ||P_n(f_{\eta,2k+2},k,t)-f_{\eta,2k+2}(t)||_{L_p(I_2)} + ||f_{\eta,2k+2}(t)-f(t)||_{L_p(I_2)}$$

$$(2.2.12) \quad = J_1 + J_2 + J_3, \text{ say.}$$

We choose numbers a^* and b^* such that $a_1 < a^* < a_2 < b_2 < b^* < b_1$. Let $\chi(u)$ be the characteristic function of $[a^*,b^*]$.

Using (1.5.2) and Lemmas 1.7.8 and 1.7.9, we obtain an estimate of J_1.

$$J_1 \leq ||P_n(\chi(f-f_{n,2k+2}),k,t)||_{L_p(I_2)} + ||P_n((1-\chi)(f-f_{n,2k+2}),k,t)||_{L_p(I_2)}$$

$$\leq M_1\{||f-f_{n,2k+2}||_{L_p[a^*,b^*]} + n^{-(k+1)}||f-f_{n,2k+2}||_{L_p(I)}\}.$$

In view of Lemma 1.3.1 this can be further bounded as

(2.2.13) $$J_1 \leq M_2\{\omega_{2k+2}(f,\eta,p,I_1) + n^{-(k+1)}||f||_{L_p(I)}\}.$$

Next, it follows from Theorems 2.2.1, 2.2.4 respectively for the cases $p > 1$ and $p = 1$, Lemma 1.2.2 and the fact

$$||f^{(2k+2)}_{n,2k+2}||_{L_1[a^*,b^*]} = ||f^{(2k+1)}_{n,2k+2}||_{B.V.[a^*,b^*]},$$

that,

$$J_2 \leq \frac{M_3}{n^{k+1}}\{||f^{(2k+2)}_{n,2k+2}||_{L_p[a^*,b^*]} + ||f_{n,2k+2}||_{L_p(I)}\}.$$

This, in conjunction with the estimates (1.3.2) and (1.3.4), implies that

(2.2.14) $$J_2 \leq \frac{M_4}{n^{k+1}}\{\eta^{-(2k+2)}\omega_{2k+2}(f,\eta,p,I_1)+||f||_{L_p(I)}\}.$$

Also, by (1.3.3) of Lemma 1.3.1

(2.2.15) $$J_3 \leq M_5\,\omega_{2k+2}(f,\eta,p,[a^*,b^*]).$$

Finally, taking $\eta = n^{-1/2}$ and (2.2.12) to (2.2.15) combined complete the proof.

Theorem 2.2.8. Let $f \in C^{2k+2}(I)$. Then there holds

$$(2.2.16) \quad P_n(f,k,t)-f(t) = n^{-(k+1)}\{\sum_{i=1}^{2k+2} Q(i,k,t)f^{(i)}(t)\} + o(n^{-(k+1)}), \ (n \to \infty),$$

uniformly in $t \in I$, where $Q(i,k,t)$'s are certain polynomials in t.

Proof. For some ξ lying between u and t we have

$$(2.2.17) \quad f(u) = \sum_{i=0}^{2k+2} \frac{(u-t)^i}{i!} f^{(i)}(t)+ \frac{(u-t)^{2k+2}}{(2k+2)!} \times (f^{(2k+2)}(\xi)-f^{(2k+2)}(t)).$$

Since $f^{(2k+2)} \in C(I)$, given an arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|f^{(2k+2)}(x) - f^{(2k+2)}(y)| < \varepsilon$, whenever $|x-y| < \delta$.

Therefore, for all u,t belonging to I

$$|(u-t)^{2k+2}(f^{(2k+2)}(\xi)-f^{(2k+2)}(t))| \le \varepsilon\ (u-t)^{2k+2} + \frac{2}{\delta^2} ||f^{(2k+2)}||_{C(I)}(u-t)^{2k+4}.$$

This implies, by positivity of $P_n(\cdot,t)$ and Corollary 1.7.7, that

$$(2.2.18) \quad |P_n((u-t)^{2k+2}(f^{(2k+2)}(\xi)-f^{(2k+2)}(t)),t)| \le \frac{M}{n^{k+1}} (\varepsilon + \frac{1}{n}).$$

From Corollary 1.7.3 and the fact that $\sum_{j=0}^{m} c(j,k)d_j^{-m} = 0$, $m = 1,2,\ldots,k$, it follows that

$$(2.2.19) \quad P_n(\frac{(u-t)^i}{i!},k,t) = n^{-(k+1)}Q(i,k,t)+o(n^{-(k+1)}), \ n \to \infty,$$

where $i = 1,2,\ldots,2k+2$ and the o-term holds uniformly in $t \in I$.

Applying the operator $P_n(\cdot,k,t)$ to (2.2.17), we obtain

$$P_n(f,k,t)-f(t) = \sum_{i=1}^{2k+2} \frac{f^{(i)}(t)}{i!} P_n((u-t)^i,k,t)$$

(2.2.20) $$+ \frac{1}{(2k+2)!} P_n((u-t)^{2k+2}(f^{(2k+2)}(\xi)-f^{(2k+2)}(t)),k,t)$$

Since $\epsilon > 0$ is arbitrary, we obtain from (2.2.18), (2.2.19) and (2.2.20)

$$P_n(f,k,t)-f(t) = n^{-(k+1)}\{\sum_{i=1}^{2k+2} Q(i,k,t)f^{(i)}(t)\} + o(n^{-(k+1)}),$$

as $n \to \infty$, where the o-term holds uniformly in $t \in I$.

2.3 INVERSE THEOREM

In view of Theorem 2.2.7 it follows that if $1 \leq p < \infty$, $f \in L_p(I)$, α be a positive number $\leq 2k+2$ and

$$\omega_{2k+2}(f,\tau,p,I_1) = O(\tau^{\alpha}) \text{ as } \tau \to 0, \text{ then}$$

$$||P_n(f,k,t)-f(t)||_{L_p(I_2)} = O(n^{-\alpha/2}) \text{ as } n \to \infty.$$

Here we prove a corresponding local inverse theorem over contracting intervals. In the proof of this theorem we require some auxiliary results which we prove first. Lemmas 2.3.2 and 2.3.3 are proved by using earlier results. Lemma 2.3.4 is a Bernstein-type inequality for Bernstein-Kantorovitch polynomials in L_p-norm. This is proved by making use of the Riesz-Thorin interpolation theorem. Finally we prove the inverse theorem by induction on α. It may be observed that in proving the inverse theorem without any loss of generality we may assume

that the function has a compact support contained in the interval $(0,1)$. To prove this, let a,b be such that $0 < a < a_1 < b_1 < b < 1$. We choose $g \in C_0^\infty$ such that $g(x) = 1$ for $x \in [a,b]$ and $\operatorname{supp} g \subset (0,1)$. Then, by Lemma 1.7.9

$$\begin{aligned} ||P_n(fg,k,t) - (fg)(t)||_{L_p(I_1)} &\le ||P_n(f,k,t)-f(t)||_{L_p(I_1)} + ||P_n(fg-f,k,t)||_{L_p(I_1)} \\ &= ||P_n(f,k,t)-f(t)||_{L_p(I_1)} + o(n^{-\ell})||f||_{L_p(I_1)}, \quad (n\to\infty), \end{aligned}$$

where ℓ is any fixed but arbitrary positive number. Thus, otherwise, instead of f we may work with fg which has a compact support contained in $(0,1)$.

Theorem 2.3.1. Let $0 < \alpha < 2k+2$, $1 \le p < \infty$ and $f \in L_p(I)$. Then

$$(2.3.1) \qquad ||P_n(f,k,t)-f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}), \quad (n \to \infty),$$

implies that

$$(2.3.2) \qquad \omega_{2k+2}(f,\tau,p,I_2) = O(\tau^\alpha), \quad (\tau \to 0).$$

Lemma 2.3.2. Let $1 \le p < \infty$, $h \in L_p(I)$ and $i,j \in \mathbb{N}^0$. Then for a fixed positive number ℓ there holds for some constant M

$$(2.3.3) \qquad ||(n+1) \sum_{\nu=0}^{n} \{ p_{n\nu}(t)|\tfrac{\nu}{n} - t|^i \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |u-t|^j \times$$

$$\times \left|\int_t^u |h(w)|dw\right|du \}||_{L_p(I_2)}$$

$$\leq M\{n^{-(i+j+1)/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p(I)}\}.$$

Proof of the Lemma. We first consider the case $p > 1$. Making use of Holder's inequality and Corollary 1.7.6

$$\sum_{\nu=0}^{n}\{p_{n\nu}(t)|\tfrac{\nu}{n}-t|^{i}\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}(n+1)|u-t|^{j}|\int_{t}^{u}|h(w)|dw|du\}$$

$$\leq \{\sum_{\nu=0}^{n} p_{n\nu}(t)|\tfrac{\nu}{n}-t|^{iq}\}^{1/q} \times$$

$$\times \{\sum_{\nu=0}^{n} p_{n\nu}(t)\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}(n+1)|u-t|^{j}|\int_{t}^{u}|h(w)|dw|du\}^{p}\}^{1/p}$$

$$\leq M_1 n^{-i/2}\{\sum_{\nu=0}^{n} p_{n\nu}(t)\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}(n+1)|u-t|^{j} \times$$

$$\times |\int_{t}^{u}|h(w)|dw|du\}^{p}\}^{1/p}.$$

Using Jensen's inequality twice the above expression is bounded by

$$M_1 n^{-i/2}\{\sum_{\nu=0}^{n} p_{n\nu}(t)\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}(n+1)|u-t|^{jp}|\int_{t}^{u}|h(w)|dw|^{p}du\}\}^{1/p}$$

$$\leq M_1 n^{-i/2}\{\sum_{\nu=0}^{n} p_{n\nu}(t)\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}(n+1)|u-t|^{(j+1)p-1} \times$$

$$\times |\int_{t}^{u}|h(w)|^{p}\,dw|du\}\}^{1/p}$$

$$= M_1\, n^{-i/2}\{P_n(|u-t|^{(j+1)p-1}|\int_{t}^{u}|h(w)|^{p}\,dw|,t)\}^{1/p}.$$

Now, proceeding as in the proof of Proposition 2.2.2 we obtain (2.3.3).

For the case $p = 1$, proceeding along the lines of proof of Proposition 2.2.5, we have to estimate the expression

$$(n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t)\left|\frac{\nu}{n}-t\right|^{i}\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}|u-t|^{r}\,du\}$$ in place of

$\int_{0}^{1} K(n,t,u)|u-t|^{r}\,du$ occuring in equation (2.2.8).

This is done by using Holder's inequality and the moment estimates for Bernstein polynomial and Bernstein-Kantorovitch (Corollaries 1.7.6 and 1.7.7) :

$$\sum_{\nu=0}^{n} p_{n\nu}(t)\left|\frac{\nu}{n}-t\right|^{i}\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}(n+1)|u-t|^{r}\,du\} \tag{2.3.4}$$

$$\leq \{\sum_{\nu=0}^{n} p_{n\nu}(t)\left|\frac{\nu}{n}-t\right|^{ip}\}^{1/p}\{\sum_{\nu=0}^{n} p_{n\nu}(t)\times$$

$$\times\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}(n+1)|u-t|^{r}\,du\}^{q}\}^{1/q}$$

$$\leq M_2 n^{-i/2}\{\sum_{\nu=0}^{n} p_{n\nu}(t)\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}(n+1)|u-t|^{rq}\,du\}^{1/q}$$

$$\leq \frac{M_2'}{n^{(r+i)/2}}\,.$$

Lemma 2.3.3. Let $1 \leq p < \infty$, $h \in L_p(I)$, and $i \in \mathbb{N}^0$. Then, for any fixed positive number ℓ, there holds

$$\|P_n(|u-t|^{i}|h(u)|,t)\|_{L_p(I_2)}$$

$$\leq M\{\frac{1}{n^{i/2}}\|h\|_{L_p(I_1)} + \frac{1}{n^{\ell}}\|h\|_{L_p(I)}\}, \tag{2.3.5}$$

where M is a constant.

Proof. Let $\chi(u)$ be the characteristic function of I_1. Using Jensen's inequality

$$|P_n(|u-t|^i|h(u)|,t)|^p = |\int_0^1 K(n,t,u)|u-t|^i|h(u)|du|^p$$

$$\leq \int_0^1 K(n,t,u)|u-t|^{ip}|h(u)|^p\,du$$

$$= \int_0^1 K(n,t,u)\chi(u)|u-t|^{ip}|h(u)|^p\,du$$

$$+\int_0^1 K(n,t,u)(1-\chi(u))|u-t|^{ip}|h(u)|^p\,du$$

$$(2.3.6) \qquad = J_1(t)+J_2(t), \text{ say.}$$

It follows from Lemma 1.7.9 that

$$(2.3.7) \qquad \left(\int_{a_2}^{b_2} J_2(t)dt\right)^{1/p} = O\left(\frac{1}{n^{\ell}}\right) ||h||_{L_p(I)}, \; (n \to \infty).$$

Now, by Fubini's theorem we have

$$\int_{a_2}^{b_2} J_1(t)dt = (n+1)\left\{\sum_{\nu=0}^{n} \int_{a_2}^{b_2} p_{n\nu}(t) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |h(u)|^p|u-t|^{ip} \times \right.$$

$$\left. \times \; \chi(u)\,du\,dt\right\}$$

$$= (n+1)\left\{\sum_{\nu=0}^{n} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |h(u)|^p\chi(u)\int_{a_2}^{b_2} p_{n\nu}(t)|u-t|^{ip}\,dt\,du\right\}$$

By mean value theorem, for some $u_o \in \left[\frac{\nu}{n+1}, \frac{\nu+1}{n+1}\right] \cap [a_2,b_2]$, the above expression becomes

$$(n+1)\left\{\sum_{\nu=0}^{n} \left(\int_{a_2}^{b_2} p_{n\nu}(t)|u_o-t|^{ip}dt\right)\left(\int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |h(u)|^p\,\chi(u)du\right)\right\}$$

It follows from (2.1.4) and the fact $\int_0^1 p_{n\nu}(t)dt = \frac{1}{n+1}$, that for any integer $r > ip$

$$\int_0^1 p_{n\nu}(t)|u_o-t|^{ip}dt \leq \frac{M_1}{n^{1-\frac{ip}{r}}}\left\{\int_0^1 p_{n\nu}(t)|u_o-t|^r\,dt\right\}^{ip/r}$$

$$\leq \frac{M_1'}{n^{1+\frac{ip}{2}}} .$$

Therefore,

(2.3.8) $$\int_{a_2}^{b_2} J_1(t)dt \leq \frac{M_2}{n^{ip/2}} ||h||^p_{L_p(I_1)} .$$

Now the lemma follows from (2.3.6), (2.3.7) and (2.3.8).

Lemma 2.3.4. Let $1 \leq p < \infty$ and $h \in L_p(I)$ with $\operatorname{supp} h \subset I_2$ Then

(2.3.9) $$||P_n^{(2k+2)}(h,t)||_{L_p(I_2)} \leq M\, n^{k+1} ||h||_{L_p(I_2)} ,$$

M being a constant independent of n and h.

If, in addition, h has 2k+2 derivatives with $h^{(2k+1)} \in A.C.(I_2)$ and $h^{(2k+2)} \in L_p(I_2)$, then

(2.3.10) $$||P_n^{(2k+2)}(h,t)||_{L_p(I_2)} \leq M_1 ||h^{(2k+2)}||_{L_p(I_2)} ,$$

the constant M_1 being independent of n and h.

Proof. By Lemma 1.7.4

$$P_n^{(2k+2)}(h,t) = (n+1)\{\sum_{\nu=0}^{n} \{\sum_{i,j} q_{ij}^{(2k+2)}(t) n^{i+j}\, p_{n\nu}(t)(\tfrac{\nu}{n} - t)^j\} \times$$

$$\times \{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)} h(u)\, du\}\} T^{-(2k+2)}$$

where i,j are non-negative integers satisfying $2i+j \leq 2k+2$.

First we obtain a bound for $P_n^{(2k+2)}(h,t)$ in $L_1(I_2)$ norm.

Using boundedness of $T^{-(2k+2)}(t)$ on I_2 and Fubini's theorem we have

$$||P_n^{(2k+2)}(h,t)||_{L_1(I_2)} \leq M_2(n+1)\{\sum_{i,j} n^{i+j}\{\sum_{\nu=0}^{n}\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}|h(u)|\times$$

$$\times(\int_0^1 p_{n\nu}(t)|\tfrac{\nu}{n}-t|^j\,dt)du\}\}.$$

Since supp $h \subset I_2$, it follows from Corollary 2.1.2 that

$$(2.3.11) \qquad ||P_n^{(2k+2)}(h,t)||_{L_1(I_2)} \leq M_2' \, n^{k+1}||h||_{L_1(I_2)} .$$

For $p = \infty$ we obtain after applying Corollary 2.1.2

$$(2.3.12) \qquad ||P_n^{(2k+2)}(h,t)||_{L_\infty(I_2)} \leq M_3 n^{k+1}||h||_{L_\infty(I_2)}.$$

Writing $M = \max(M_2', M_3)$, (2.3.9) follows from (2.3.11), (2.3.12) and the Riesz-Thorin interpolation theorem (Lemma 1.2.3).

To prove (2.3.10), for $u,t \in I_2$, we write

$$h(u) = \sum_{i=0}^{2k+1}\frac{(u-t)^i}{i!}h^{(i)}(t) + \frac{1}{(2k+1)!}\int_t^u (u-w)^{2k+1}h^{(2k+2)}(w)dw,$$

so that

$$P_n(h,x) = \sum_{i=0}^{2k+1}\frac{h^{(i)}(t)}{i!}P_n((u-t)^i,x)$$

$$+ \frac{1}{(2k+1)!}P_n(\int_t^u (u-w)^{2k+1}h^{(2k+2)}(w)\,dw,\ x).$$

Since $P_n(\cdot,x)$ maps algebraic polynomials into algebraic polynomials of same degree,

$$P_n^{(2k+2)}(h,t) = \frac{1}{(2k+1)!}P_n^{(2k+2)}(\int_t^u (u-w)^{2k+1}h^{(2k+2)}(w)dw,\ t)$$

$$= \frac{1}{(2k+1)!}\{\sum_{i,j} n^{i+j}q_{ij}^{(2k+2)}(t)\{(n+1)\{\sum_{\nu=0}^{n}p_{n\nu}(t)(\tfrac{\nu}{n}-t)^j\times$$

$$\times\, T^{-(2k+2)}\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)} \int_t^u (u-w)^{2k+1} h^{(2k+2)}(w)dw\, du\}\}\}\}.$$

Hence by Lemma 2.3.2

$$||P_n^{(2k+2)}(h,t)||_{L_p(I_2)} \leq M_1||h^{(2k+2)}||_{L_p(I_2)},$$

completing the proof.

Proof of Theorem 2.3.1. Let (x_i,y_i), $i = 1,2,3$ satisfy $a_1 < x_1 < x_2 < x_3 < a_2 < b_2 < y_3 < y_2 < y_1 < b_1$. We choose a function $g \in C_o^{2k+2}$ such that $\operatorname{supp} g \subset (x_2,y_2)$ and $g(t) = 1$ for $t \in [x_3,y_3]$. Writing $fg = \bar{f}$, for all values of $\gamma \leq \tau$ we have

$$(2.3.13)\qquad ||\Delta_\gamma^{2k+2}\, \bar{f}(t)||_{L_p[x_2,y_2]}$$

$$\leq ||\Delta_\gamma^{2k+2}\{\bar{f}(t)-P_n(\bar{f},k,t)\}||_{L_p[x_2,y_2]}$$

$$+\, ||\Delta_\gamma^{2k+2}\, P_n(\bar{f},k,t)||_{L_p[x_2,y_2]}.$$

Now, by Lemma 1.1.2

$$||\Delta_\gamma^{2k+2}\, P_n(\bar{f},k,t)||_{L_p[x_2,y_2]}$$

$$= ||\int_0^\gamma \cdots \int_0^\gamma P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i)dz_1 \ldots dz_{2k+2}||_{L_p[x_2,y_2]}.$$

Now by Jensen's inequality

$$|\int_0^\gamma \cdots \int_0^\gamma P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i)dz_1 \ldots dz_{2k+2}|^p$$

$$\leq \gamma^{p-1} \int_0^\gamma \{ \int_0^\gamma \cdots \int_0^\gamma |P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i)| dz_1 \cdots dz_{2k+1} \}^p \, dz_{2k+1} .$$

Thus we see from here that repeated application of Jensen's inequality (2k+2 times) gives

$$|\int_0^\gamma \cdots \int_0^\gamma P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i) \, dz_1 \cdots dz_{2k+2}|^p$$

$$\leq \gamma^{(2k+2)(p-1)} \int_0^\gamma \cdots \int_0^\gamma |P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i)|^p \, dz_1 \cdots dz_{2k+2} .$$

Also using Fubini's theorem repeatedly we obtain

$$\int_{x_2}^{y_2} |\int_0^\gamma \cdots \int_0^\gamma P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i) \, dz_1 \cdots dz_{2k+2}|^p \, dt$$

$$\leq \gamma^{(2k+2)(p-1)} \times$$

$$\times \int_0^\gamma \cdots \int_0^\gamma \int_{x_2}^{y_2} |P_n^{(2k+2)}(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i)|^p \, dt \, dz_1 \cdots dz_{2k+2}$$

$$\leq \gamma^{(2k+2)p} \, ||P_n^{(2k+2)}(\bar{f},k,t)||^p_{L_p[x_2',y_2']},$$

where $x_2' = x_2$ and $y_2' = y_2 + (2k+2)\gamma$.

Hence

$$||\Delta_\gamma^{2k+2} P_n(\bar{f},k,t)||_{L_p[x_2,y_2]}$$

$$\leq \gamma^{(2k+2)} ||P_n^{(2k+2)}(\bar{f},k,t)||_{L_p[x_2',y_2']}$$

$$\leq \gamma^{2k+2} \{ ||P_n^{(2k+2)}(\bar{f}-\bar{f}_{n,2k+2},k,t)||_{L_p[x_2',y_2']}$$

$$+ ||P_n^{(2k+2)}(\bar{f}_{n,2k+2},k,t)||_{L_p[x_2',y_2']} \}.$$

Hence Lemma 2.3.4 implies that for all sufficiently small $\eta > 0$

$$||\Delta_\gamma^{2k+2} P_n(\bar{f},k,t)||_{L_p[x_2,y_2]}$$

$$\leq M_4 \gamma^{2k+2}\{n^{k+1}||\bar{f}-\bar{f}_{\eta,2k+2}||_{L_p[x_2,y_2]}+||\bar{f}^{(2k+2)}_{\eta,2k+2}||_{L_p[x_2,y_2]}\}.$$

This, in conjunction with the estimates obtained in (1.3.2) and (1.3.3), gives

$$(2.3.14) \qquad ||\Delta_\gamma^{2k+2} P_n(\bar{f},k,t)||_{L_p[x_2,y_2]}$$

$$\leq M_4' \gamma^{2k+2}(n^{k+1}+\frac{1}{\eta^{2k+2}}) \omega_{2k+2}(\bar{f},\eta,p,[x_2,y_2]).$$

The next major step is to show that

$$(2.3.15) \quad ||\Delta_\gamma^{2k+2}\{\bar{f}(t)-P_n(\bar{f},k,t)\}||_{L_p[x_2,y_2]}=O(n^{-\alpha/2})(n \to \infty).$$

For, after having proved (2.3.15) we may combine (2.3.13), (2.3.14) and (2.3.15) to get

$$||\Delta_\gamma^{2k+2} \bar{f}(t)||_{L_p[x_2,y_2]}$$

$$\leq M_5 \{\frac{1}{n^{\alpha/2}} + \gamma^{2k+2}(n^{k+1}+\frac{1}{\eta^{2k+2}})\omega_{2k+2}(\bar{f},\eta,p,[x_2,y_2])\} .$$

Then choosing η such that $n \leq \eta^{-2} < 2n$, we obtain

$$||\Delta_\gamma^{2k+2} \bar{f}(t)||_{L_p[x_2,y_2]} \leq 2M_5\{\eta^{\alpha}+(\frac{\tau}{\eta})^{2k+2} \omega_{2k+2}(\bar{f},\eta,p,[x_2,y_2])\}.$$

Since this holds for all $\gamma \leq \tau$, we have

$$\omega_{2k+2}(\bar{f},\tau,p,[x_2,y_2]) \leq 2M_5\{\eta^{\alpha}+(\frac{\tau}{\eta})^{2k+2}\,\omega_{2k+2}(f,\eta,p,[x_2,y_2])\}.$$

This implies, by Lemma 1.3.7, that

$$\omega_{2k+2}(\bar{f},\tau,p,[x_2,y_2]) = O(\tau^{\alpha}),\ (\tau \to 0).$$

Therefore, as $\bar{f}(t) = f(t)$ for $t \in [x_3,y_3]$

$$\omega_{2k+2}(f,\tau,p,I_2) = O(\tau^{\alpha}),\ (\tau \to 0),$$

as required.

We prove (2.3.15) by induction on α. Consider first the case when $\alpha \leq 1$.

$$\begin{aligned} &||P_n(fg,k,t)-(fg)(t)||_{L_p[x_2,y_2]} \\ &\quad \leq ||P_n((f(u)-f(t))\,g(t),k,t)||_{L_p[x_2,y_2]} \\ &\qquad\qquad + ||P_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_2,y_2]} \\ &\quad = ||g(t)\{P_n(f,k,t)-f(t)\}||_{L_p[x_2,y_2]} \\ &\qquad\qquad + ||P_n(f(u)(u-t)g'(\xi),k,t)||_{L_p[x_2,y_2]}, \end{aligned}$$

for some ξ lying between u and t. We use Lemma 2.3.3 to obtain a bound for the second term. This alongwith (2.3.1) gives

$$||P_n(fg,k,t)-(fg)(t)||_{L_p[x_2,y_2]} \leq \frac{M_6}{n^{\alpha/2}} + \frac{M_6}{n^{1/2}} \leq \frac{2M_6}{n^{\alpha/2}}\ .$$

This proves (2.3.15) when $\alpha \leq 1$.

Now we assume that for some $r \leq 2k+1$, the theorem holds for all values of α satisfying $r-1 \leq \alpha < r$. We are then to show that the theorem also remains valid for all α satisfying $r \leq \alpha < r+1$. For this if $f_{\eta,2k+2}(u)$ is the Steklov mean of $(2k+2)$th order corresponding to $f(u)$,

$$||P_n(fg,k,t)-(fg)(t)||_{L_p[x_2,y_2]}$$

$$\leq ||P_n((f(u)-f(t))g(t),k,t)||_{L_p[x_2,y_2]}$$

$$+ ||P_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_2,y_2]}$$

$$\leq \frac{M_6'}{n^{\alpha/2}} + ||P_n((f(u)-f_{\eta,2k+2}(u))(g(u)-g(t)),k,t)||_{L_p[x_2,y_2]}$$

$$+ ||P_n((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t))(g(u)-g(t)),k,t)||_{L_p[x_2,y_2]}$$

$$+ ||P_n(f_{\eta,2k+2}(t)(g(u)-g(t)),k,t)||_{L_p[x_2,y_2]}$$

$$(2.3.16) \qquad = \frac{M_6'}{n^{\alpha/2}} + J_1 + J_2 + J_3, \text{ say.}$$

By Theorem 2.2.8 and (1.3.4) of Lemma 1.3.1

$$(2.3.17) \qquad J_3 \leq \frac{M_7}{n^{k+1}} .$$

Moreover, for some ξ lying between u and t,

$$J_1 = ||P_n((f(u)-f_{\eta,2k+2}(u))(u-t)g'(\xi),k,t)||_{L_p[x_2,y_2]}$$

$$\le \|g'\|_{C(I)} \{ \sum_{j=0}^{k} |c(j,k)| \times$$

$$\times \{ \| P_{djn}(|f(u)-f_{n,2k+2}(u)|\cdot|u-t|,t) \|_{L_p[x_2,y_2]} \}\}.$$

Hence, by (1.3.3), (1.3.4) and Lemma 2.3.3, for any positive number ℓ

$$(2.3.18) \quad J_1 \le M_7' \{ n^{-1/2} \|f-f_{n,2k+2}\|_{L_p[x_1,y_1]} + n^{-\ell} \|f\|_{L_p(I)} \}.$$

We have for some ξ lying between u and t

$$(f_{n,2k+2}(u)-f_{n,2k+2}(t))(g(u)-g(t))$$

$$= \{ \sum_{i=1}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}_{n,2k+2}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f^{(2k+2)}_{n,2k+2}(w)dw \} \times$$

$$\times \{ \sum_{i=1}^{2k} \frac{(u-t)^i}{i!} g^{(i)}(t) + \frac{(u-t)^{2k+1}}{(2k+1)!} g^{(2k+1)}(\xi) \}$$

$$= \frac{1}{(2k+1)!} \{ \sum_{i=1}^{2k} \frac{g^{(i)}(t)}{i!} (u-t)^i \{ \int_t^u (u-w)^{2k+1} f^{(2k+2)}_{n,2k+2}(w)dw \}\}$$

$$+ \frac{1}{((2k+1)!)^2} \{ g^{(2k+2)}(\xi)(u-t)^{2k+1} \int_t^u (u-w)^{2k+1} f^{(2k+2)}_{n,2k+2}(w)dw \}$$

$$+ \sum_{i=1}^{2k+1} \sum_{j=1}^{2k} \{ \frac{f^{(i)}_{n,2k+2}(t) g^{(j)}(t)}{i!j!} (u-t)^{i+j} \}$$

$$+ \frac{g^{(2k+1)}(\xi)}{(2k+1)!} \{ \sum_{i=1}^{2k+1} \frac{(u-t)^{2k+1+i}}{i!} f^{(i)}_{n,2k+2}(t) \} .$$

Therefore,

$$J_2 \le \frac{1}{(2k+1)!} \| P_n ((\sum_{i=1}^{2k} \frac{g^{(i)}(t)}{i!} (u-t)^i) \times$$

$$\times (\int_t^u (u-w)^{2k+1} f^{(2k+2)}_{n,2k+2}(w)dw), k, t) \|_{L_p[x_2,y_2]} +$$

$$+ \frac{1}{((2k+1)!)^2} \| P_n (g^{(2k+2)}(\xi)(u-t)^{2k+1} \times$$

$$\times \int_t^u (u-w)^{2k+1} f_{n,2k+2}^{(2k+2)}(w)dw, k, t) \|_{L_p[x_2,y_2]}$$

$$+ \{ \sum_{i=1}^{2k+1} \sum_{j=1}^{2k} \frac{1}{i!\,j!} \| f_{n,2k+2}^{(i)}(t) g^{(j)}(t) P_n((u-t)^{i+j}, k, t) \|_{L_p[x_2,y_2]} \}$$

$$+ \frac{1}{(2k+1)!} \{ \sum_{i=1}^{2k+1} \frac{1}{i!} \| f_{n,2k+2}^{(i)}(t) \times$$

$$\times P_n((u-t)^{2k+1+i} g^{(2k+1)}(\xi), k, t) \|_{L_p[x_2,y_2]} \}$$

(2.3.19) $\quad = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$, say.

By Lemma 2.3.2, for any fixed positive number ℓ ,

(2.3.20) $$\Sigma_1 \le M_8 \{ \sum_{i=1}^{2k} \frac{1}{n^{k+1+\frac{i}{2}}} \| f_{n,2k+2}^{(2k+2)} \|_{L_p[x_1,y_1]} + \frac{1}{n^{\ell}} \| f_{n,2k+2}^{(2k+2)} \|_{L_p(I)} \}$$

and

(2.3.21) $$\Sigma_2 \le M_8' \{ \frac{1}{n^{2k+\frac{3}{2}}} \| f_{n,2k+2}^{(2k+2)} \|_{L_p[x_1,y_1]} + \frac{1}{n^{\ell}} \| f_{n,2k+2}^{(2k+2)} \|_{L_p(I)} \}.$$

It follows from Corollary 1.7.3 and the fact

$\sum_{j=0}^{k} c(f,k)\, d_j^{-m} = 0$, $m = 1,2,\ldots,k$, that

$$\Sigma_3 \leq \frac{M_8'}{n^{k+1}} \left(\sum_{i=1}^{2k+1} ||f_{\eta,2k+2}^{(i)}||_{L_p[x_2,y_2]} \right).$$

Hence by Lemma 1.2.2

(2.3.22) $$\Sigma_3 \leq \frac{M_9}{n^{k+1}} \left(||f_{\eta,2k+2}^{(2k+1)}||_{L_p[x_2,y_2]} + ||f_{\eta,2k+2}||_{L_p[x_2,y_2]} \right).$$

Clearly $\Sigma_4 \leq \frac{M_9'}{n^{k+1}} \left(\sum_{i=1}^{2k+1} ||f_{\eta,2k+2}^{(i)}||_{L_p[x_2,y_2]} \right).$

Using estimates (1.2.3) of Lemma 1.2.2 we have

(2.3.23) $$\Sigma_4 \leq \frac{M_{10}}{n^{k+1}} \left(||f_{\eta,2k+2}^{(2k+1)}||_{L_p[x_2,y_2]} + ||f_{\eta,2k+2}||_{L_p[x_2,y_2]} \right).$$

If we choose points c,d such that $a_1 < c < x_1 < y_1 < d < b_1$, then by induction hypothesis we can assume that

(2.3.24) $$\omega_{2k+2}(f,\eta,p,[c,d]) = O(\eta^{\alpha-1}), \ (\eta \to 0).$$

This implies by Corollary 1.3.4 that also

(2.3.25) $$\omega_{2k+1}(f,\eta,p,[c,d]) = O(\eta^{\alpha-1}), \ (\eta \to 0).$$

On taking $\ell = 2k+2$ and $\eta = n^{-1/2}$, it follows from (2.3.20), (2.3.21) and estimates (1.3.2) to (1.3.5) obtained in Lemma 1.3.1 that

(2.3.26) $$\Sigma_1, \Sigma_2 \leq M_{10}' \left\{ \frac{1}{n^{1/2}} \omega_{2k+2}(f,n^{-1/2},p,[c,d]) + \frac{1}{n^{k+1}} ||f||_{L_p(I)} \right\}.$$

And from (2.3.22) and (2.3.23)

$$\Sigma_3, \Sigma_4 \leq M_{11}\{ \frac{1}{n^{1/2}} \omega_{2k+1}(f, n^{-1/2}, p, [c,d]) + \frac{1}{n^{k+1}} ||f||_{L_p(I)} \}. \tag{2.3.27}$$

Thus we see from (2.3.24), (2.3.25), (2.3.26) and (2.3.27) upon taking $\eta = n^{-1/2}$ that

$$\Sigma_1,\ \Sigma_2,\ \Sigma_3 \text{ and } \Sigma_4 \leq \frac{M_{12}}{n^{\alpha/2}}\,. \tag{2.3.28}$$

Also, taking $\eta = n^{-1/2}$ and $\ell = k+1$ we have from (2.3.18) and (2.3.24)

$$J_1 \leq \frac{M_{13}}{n^{\alpha/2}}\,. \tag{2.3.29}$$

Finally from (2.3.16) and the estimates of J_1, J_2 and J_3 obtained in (2.3.17), (2.3.19), (2.3.28) and (2.3.29) we conclude that

$$||P_n(fg,k,t)-(fg)(t)||_{L_p[x_2,y_2]} = O(n^{-\alpha/2}),\ (n \to \infty).$$

From this follows (2.3.15) and hence the proof of the theorem.

2.4 SATURATION THEOREM

In this section we prove that in L_p-approximation where $1 \leq p < \infty$, the sequence $\{P_n(\cdot,k,t)\}$ is saturated with the order $O(n^{-(k+1)})$. The nature of the saturation class depends on whether $p = 1$ or $p > 1$. The trivial class, however, remains the same for all $p(1 \leq p < \infty)$. The theorem is in a local set-up over contracting intervals.

Theorem 2.4.1. Let $1 \leq p < \infty$ and $f \in L_p(I)$. Then, in the following, the implications "(i) ==> (ii) ==> (iii)" and "(iv) ==> (v) ==> (vi)" hold.

(i) $||P_n(f,k,t)-f(t)||_{L_p(I_1)} = O(n^{-(k+1)}), \quad (n \to \infty);$

(ii) f coincides a.e. with a function F on I_2 having 2k+2 derivatives such that (a) when $p > 1$, $F^{(2k+1)} \in A.C.(I_2)$ and $F^{(2k+2)} \in L_p(I_2)$ and (b) when $p = 1$, $F^{(2k)} \in A.C.(I_2)$ and $F^{(2k+1)} \in B.V.(I_2)$;

(iii) $||P_n(f,k,t)-f(t)||_{L_p(I_3)} = O(n^{-(k+1)}), \quad (n \to \infty);$

(iv) $||P_n(f,k,t)-f(t)||_{L_p(I_1)} = o(n^{-(k+1)}), \quad (n \to \infty);$

(v) f coincides a.e. with a function F on I_2, where F is 2k+2 times continuously differentiable on I_2 and satisfies $\sum_{j=1}^{2k+2} Q(j,k,t)\, F^{(j)}(t) = 0$, where $Q(j,k,t)$ are the polynomials occuring in (2.2.16);

(vi) $||P_n(f,k,t)-f(t)||_{L_p(I_3)} = o(n^{-(k+1)}), \quad (n \to \infty).$

To prove the theorem we need the following lemma.

Lemma 2.4.2. Let $1 \leq p < \infty$, $h \in L_p(I)$ with supp $h \subset (0,1)$. If h has 2k+1 derivatives with $h^{(2k)} \in A.C.(I)$ and $h^{(2k+1)} \in L_p(I)$, then for each $g \in C_0^{2k+2}$ with supp $g \subset (0,1)$, there holds

$$(2.4.1) \quad |\langle P_n(h,k,t)-h(t),g(t)\rangle| \leq \frac{M}{n^{k+1}} \{ ||h^{(2k+1)}||_{L_p(I)} + ||h||_{L_p(I)} \},$$

where the constant M does not depend on h or n.

Proof. We have by Fubini's theorem

$$\langle P_n(h,t),g(t)\rangle = \int_0^1 P_n(h,t)g(t)dt = \int_0^1\int_0^1 K(n,t,u)h(u)g(t)du\,dt$$
$$= \int_0^1\int_0^1 K(n,t,u)h(u)g(t)dt\,du.$$

For some ξ lying between u and t, this reduces to

$$\int_0^1\int_0^1 K(n,t,u)h(u)\{\sum_{i=0}^{2k+1} \frac{(t-u)^i}{i!} g^{(i)}(u) + \frac{(t-u)^{2k+2}}{(2k+2)!} g^{(2k+2)}(\xi)\}dt\,du$$

$$= \sum_{i=0}^{2k+1} \{\frac{1}{i!} \int_0^1\int_0^1 K(n,t,u)(t-u)^i h(u)g^{(i)}(u)dt\,du\}$$
$$+ \frac{1}{(2k+2)!} \int_0^1\int_0^1 K(n,t,u)(t-u)^{2k+2}h(u)g^{(2k+2)}(\xi)dt\,du.$$

Writing $h_i(u) = h(u)g^{(i)}(u)$, $0 \leq i \leq 2k+1$, the above expression

$$= \sum_{i=0}^{2k+1} \frac{1}{i!} \int_0^1\int_0^1 K(n,t,u)(t-u)^i\, h_i(u)dt\,du$$
$$+ \frac{1}{(2k+2)!} \int_0^1\int_0^1 K(n,t,u)(t-u)^{2k+2}h(u)g^{(2k+2)}(\xi)dt\,du$$

(2.4.2) $$= \sum_{i=0}^{2k+1} \frac{1}{i!} J_i + \frac{1}{(2k+2)!} J_{2k+2}, \text{ say.}$$

Then,

(2.4.3) $$J_o = \int_0^1\int_0^1 K(n,t,u)h_o(u)dt\,du = \int_0^1 h_o(u)du.$$

Using the fact that supp $h \subset (0,1)$ and proceeding as in the estimate of J_1 in Lemma 2.3.3 we obtain

$$|J_{2k+2}| \leq \frac{1}{(2k+2)!}||g^{(2k+2)}||_{C(I)}\int_0^1 |h(u)|\int_0^1 K(n,t,u)(t-u)^{2k+2}dt\,du$$

(2.4.4) $$\leq \frac{M_1}{n^{k+1}} ||h||_{L_1(I)} \leq \frac{M_1}{n^{k+1}} ||h||_{L_p(I)}.$$

Next, for $1 \le i \le 2k+1$, by Fubini's theorem, we have

$$J_i = \int_0^1\int_0^1 K(n,t,u)h_i(u)(t-u)^i \, du \, dt.$$

Since $h_i(u)$ can be expanded as

$$h_i(u) = \sum_{j=0}^{2k+1-i} \frac{(u-t)^j}{j!} h_i^{(j)}(t) +$$

$$+ \frac{1}{(2k+1-i)!}\int_t^u (u-w)^{2k+1-i} h_i^{(2k+2-i)}(w) \, dw,$$

hence

$$J_i = (-1)^i \sum_{j=0}^{2k+1-i} \frac{1}{j!} \int_0^1\int_0^1 K(n,t,u)h_i^{(j)}(t)(u-t)^{i+j} \, du \, dt$$

$$+ \frac{(-1)^i}{(2k+1-i)!} \int_0^1\int_0^1 K(n,t,u)(u-t)^i \times$$

$$\times \int_t^u (u-w)^{2k+1-i} h_i^{(2k+2-i)}(w) dw \, du \, dt$$

(2.4.5) $$= (-1)^i \sum_{j=0}^{2k+1-i} \frac{1}{j!} S_{i,j} + \frac{(-1)^i}{(2k+1-i)!} S_{i,2k+2-i}, \text{ say.}$$

It follows from Lemma 2.3.2 that

(2.4.6) $$|S_{i,2k+2-i}| \le \frac{M_2}{n^{k+1}} \, ||h_i^{(2k+2-i)}||_{L_p(I)}.$$

Collecting (2.4.2) to (2.4.6) we obtain

$$\langle P_n(h,t),g(t)\rangle = \langle h(t),g(t)\rangle + \sum_{i=1}^{2k+1} \sum_{j=0}^{2k+1-i} \frac{(-1)^i}{i!\,j!} S_{i,j}$$

$$+ \left(\sum_{i=0}^{2k+1} ||h^{(i)}||_{L_p(I)}\right) O\left(\frac{1}{n^{k+1}}\right),$$

where the O-term does not depend on h.

Since, for $0 \leq j \leq 2k+1-i$,

$$S_{i,j} = \int_0^1 h_i^{(j)}(t)\, P_n((u-t)^{i+j},t)\, dt,$$

we have

$$\langle P_n(h,k,t)-h(t),g(t)\rangle = \sum_{i=1}^{2k+1} \sum_{j=0}^{2k+1-i} \frac{(-1)^i}{i!\; j!} \times$$

$$\times\left(\int_0^1 h_i^{(j)}(t)\, P_n((u-t)^{i+j},k,t)\, dt\right)$$

$$+ \left(\sum_{i=0}^{2k+1} ||h^{(i)}||_{L_p(I)}\right) O\left(\frac{1}{n^{k+1}}\right).$$

Applying Corollary 1.7.3, Lemma 1.2.2 alongwith the fact that $\sum_{j=0}^{k} c(j,k)\, d_j^{-m} = 0$, $m = 1,2,\ldots,k$, to the terms on the right side of the above expression, we obtain the inequality 2.4.1.

Proof of Theorem 2.4.1. Assume (i) of Theorem 2.4.1. Then it follows from inverse theorem (Theorem 2.3.1) and Theorem 1.3.2 that for $a_1 < c < d < b_1$, f coincides a.e. on $[c,d]$ with a function F possessing an absolutely continuous derivative $F^{(2k)}$, and a (2k+1)th derivative $F^{(2k+1)}$ which belongs to $L_p[c,d]$. Moreover, for any integer k, there holds for $0 < \beta < 1$

(2.4.7) $\quad \omega_k(F^{(2k+1)},\tau,p,[c,d]) = O(\tau^{\beta}), \ (\tau \to 0).$

We choose pairs of points x_1,x_2 and y_1,y_2 such that $a_1 < x_1 < x_2 < a_2 < b_2 < y_2 < y_1 < b_1$. Let $q \in C_0^{2k+2}$ with support

$q \subset (a_1,b_1)$ and $q(t) = 1$ for $t \in [x_1,y_1]$.

Define a function G by $G(u) = F(u)q(u)$, $u \in I$.

Then, $$||P_n(G,k,t)-G(t)||_{L_p[x_2,y_2]}$$

$$\leq ||P_n(f,k,t)-f(t)||_{L_p[x_2,y_2]}$$

$$+ ||P_n(G-f,k,t)||_{L_p[x_2,y_2]}.$$

It follows from Lemma 1.7.9 that

$$||P_n(G-f,t)||_{L_p[x_2,y_2]} = O(n^{-(k+1)}), \quad (n \to \infty),$$

and hence

$$||P_n(G-f,k,t)||_{L_p[x_2,y_2]} = O(n^{-(k+1)}), \ (n \to \infty).$$

This alongwith the hypothesis that (i) holds, implies

(2.4.8) $$||P_n(G,k,t)-G(t)||_{L_p[x_2,y_2]} = O(n^{-(k+1)}), \ (n \to \infty).$$

Now, if $p > 1$, it follows by Alaoglu's theorem (see Lemma 1.2.4) that there exists a function $H(t) \in L_p[x_2,y_2]$ such that for some subsequence $\{n_j\}$ and for every $g \in C_o^{2k+2}$ with supp $g \subset (0,1)$

(2.4.9) $$\lim_{n_j \to \infty} n_j^{k+1} \langle P_{n_j}(G,k,t)-G(t),g(t)\rangle = \langle H(t),g(t)\rangle.$$

When $p = 1$, the functions $\phi_n(x)$ defined by

(2.4.10) $$\phi_n(x) = \int_{x_2}^{x} n^{k+1} \{P_n(G,k,t)-G(t)\}\, dt$$

are, by (2.4.8) uniformly bounded and are of uniformly bounded variation. Making use of Alaoglu's theorem (Lemma 1.2.4), it follows that there exists a function $\phi_o(x)$ of bounded variation over $[x_2,y_2]$ such that for some subsequence $\{n_j\}$ and for every $g \in C_o^{2k+2}$ with supp $g \subset (x_2,y_2)$

$$(2.4.11) \qquad \int_{x_2}^{y_2} g(t)d(\phi_{n_j}(t)-\phi_o(t)) \to 0, \quad (n_j \to \infty).$$

Now, $$\int_{x_2}^{y_2} g(t)d(\phi_{n_j}(t)-\phi_o(t))$$

$$=\int_{x_2}^{y_2} g(t)\, d\phi_{n_j}(t) - \int_{x_2}^{y_2} g(t)d\phi_o(t).$$

It follows from (2.4.10), Theorem 17.17 of [30] and the fact that supp $g \subset (x_2,y_2)$ that

$$\int_{x_2}^{y_2} g(t)d(\phi_{n_j}(t)-\phi_o(t))$$
$$= n_j^{k+1}\int_{x_2}^{y_2} g(t)\{P_{n_j}(G,k,t)-G(t)\}\, dt + \int_{x_2}^{y_2} g'(t)\phi_o(t)dt.$$

This together with (2.4.11) implies tnat

$$(2.4.12) \qquad \lim_{n_j \to \infty} n_j^{k+1} \langle P_{n_j}(G,k,t)-G(t),g(t)\rangle = -\langle \phi_o(t),g'(t)\rangle.$$

As the Steklov means $G_{n,2k+2}$ for G have continuous derivatives of order upto $2k+2$, using (2.4.7) for $i = 0,1,\dots,2k+1$, there holds

$$(2.4.13) \qquad ||G^{(i)}_{n,2k+2}-G^{(i)}||_{L_p(I_1)} \to 0, \quad (n \to 0).$$

By Theorem 2.2.8

(2.4.14) $P_{n_j}(G_{n,2k+2},k,t)-G_{n,2k+2}(t)$

$$= \frac{1}{n_j^{k+1}} P_{2k+2}(D)G_{n,2k+2}(t) + o\left(\frac{1}{n_j^{k+1}}\right),$$

where $P_{2k+2}(D)G_{n,2k+2}(t) = \sum_{i=1}^{2k+2} Q(i,k,t)G^{(i)}_{n,2k+2}(t)$

and the o-term may possibly depend on n.

Hence, if $P^*_{2k+2}(D)$ denotes the differential operator adjoint to $P_{2k+2}(D)$, we have by (2.4.14)

$$\langle G_{n,2k+2}(t),P^*_{2k+2}(D)g(t)\rangle = \langle P_{2k+2}(D)G_{n,2k+2}(t),g(t)\rangle$$

$$= \lim_{n_j\to\infty} n_j^{k+1} \langle P_{n_j}(G_{n,2k+2},k,t)-G_{n,2k+2}(t),g(t)\rangle$$

$$= \lim_{n_j\to\infty} n_j^{k+1} \langle P_{n_j}(G_{n,2k+2}-G,k,t)-(G_{n,2k+2}(t)-G(t),g(t)\rangle$$

$$+ \lim_{n_j\to\infty} n_j^{k+1} \langle P_{n_j}(G,k,t)-G(t),g(t)\rangle.$$

i.e., $\langle G_{n,2k+2}(t),\ P^*_{2k+2}(D)g(t)\rangle$

$$- \lim_{n_j\to\infty} n_j^{k+1} \langle P_{n_j}(G,k,t)-G(t),g(t)\rangle$$

$$= \lim_{n_j\to\infty} n_j^{k+1} \langle P_{n_j}(G_{n,2k+2}-G,k,t)-(G_{n,2k+2}(t)-G(t),g(t)\rangle.$$

Hence, by Lemma 2.4.2

$$\langle G_{n,2k+2}(t),P^*_{2k+2}(D)g(t)\rangle - \lim_{n_j\to\infty} n_j^{k+1} \langle P_{n_j}(G,k,t)-G(t),g(t)\rangle$$

$$(2.4.15) \quad \leq M\{||G^{(2k+1)}_{n,2k+1}-G^{(2k+1)}||_{L_p(I)}+||G_{n,2k+1}-G||_{L_p(I)}\}.$$

Taking limit as $n \to 0$ in (2.4.15) and using (2.4.13) we obtain

$$(2.4.16) \quad \langle G(t),P^*_{2k+2}(D)g(t)\rangle = \lim_{n_j\to\infty} n_j^{k+1}\langle P_{n_j}(G,k,t)-G(t),g(t)\rangle.$$

Comparing (2.4.16) with (2.4.9) and (2.4.12) we have

$$(2.4.17) \quad \langle G(t),P^*_{k2k+2}(D)g(t)\rangle = \begin{cases} \langle H(t),g(t)\rangle, & \text{if } p > 1, \\ -\langle \phi_o(t),g'(t)\rangle, & \text{if } p = 1. \end{cases}$$

Using integration by parts it easily follows that

$$(2.4.18) \quad \langle G(t),P^*_{2k+2}(D)g(t)\rangle$$

$$= \langle Q(2k+2,k,t)G(t) + \sum_{i=1}^{2k+2} I_i(b_iG)(t),g^{(2k+2)}(t)\rangle,$$

where $b_i(t)$ are certain polynomials in t and I_i denotes the ith iterated indefinite integral operator, namely

$$I_i(t) = \int_o^t..\int_o^t (.)\, dt...dt.$$

Similarly

$$(2.4.19) \quad \langle H(t),g(t)\rangle = \langle I_{2k+2}(H)(t),g^{(2k+2)}(t)\rangle.$$

When $p > 1$, from (2.4.18) and (2.4.19) we have

$$(2.4.20) \quad \int_0^1 \{ Q(2k+2,k,t)G(t) + \sum_{i=1}^{2k+2} I_i(b_iG)(t) - I_{2k+2}(H)(t)\} \times$$
$$\times\, g^{(2k+2)}(t) = 0.$$

It follows from Theorem 2.2.8 and Lemma 1.7.1 that

$Q(2k+2,k,t) = c_k(t(1-t))^{k+1}$, where c_k is a nonzero constant.

This implies by Lemma 1.1.1 and the assumed smoothness hypothesis for f (stated in the beginning of this proof) that $G^{(2k+1)} \in A.C.\ [x_2,y_2]$ and $G^{(2k+2)} \in L_p[x_2,y_2]$.

Since $G(u) = F(u)$ for $u \in [x_1,y_1]$, we have $F^{(2k+1)} \in A.C.(I_2)$ and $F^{(2k+2)} \in L_p(I_2)$.

When $p = 1$, proceeding similarly, we obtain $F^{(2k+1)} \in B.V.(I_2)$.

This completes the proof of the implication "(i) ==> (ii)".

The implication "(ii) ==> (iii)" follows from Theorem 2.2.1 and 2.2.4, respectively for the cases $p > 1$ and $p = 1$.

Assuming (iv) and proceeding as in the proof of the implication "(i) ==> (ii)", we first find that $H(t)$ and $\phi(t)$ are zero functions. This does imply that F is 2k+2 times continuously differentiable function and that $P_{2k+2}(D)\, F(t) = 0$.

Finally "(v) ==> (vi)" holds by Theorem 2.2.8.

This completes the proof.

CHAPTER III

L_p-APPROXIMATION BY INTERPOLATORY MODIFICATIONS OF BERNSTEIN-KANTOROVITCH POLYNOMIALS

In Chapter II we showed that the linear combinations of Bernstein-Kantorovitch polynomials furnish an improved order of approximation in L_p-norm. Also we proved a related inverse and saturation theorems. In this chapter we show that operators $P_{n,m}(\cdot,t)$(defined in Section 6, Chapter I) may also be used to obtain a better order of approximation in L_p-norm $(1 \leq p < \infty)$. Next, we obtain corresponding inverse and saturation theorems. The proofs of these theorems make use of some estimates obtained in Chapter II. In Section 1 we establish the basic convergence of the sequence of operators $\{P_{n,m}(\cdot,t)\}$ in L_p-norm $(1 \leq p < \infty)$. In Section 2 we obtain bounds for the error in L_p-approximation by $P_{n,m}(\cdot,t)$ in terms of norms of derivatives of function and also in terms of $(m+1)$th modulus of smoothness of the function. In Section 3 we prove the inverse theorem and in Section 4 the Euler-Maclaurin sum formula with a remainder term is used to prove the saturation theorem.

3.1 BASIC APPROXIMATION

In this section we first obtain a formula which expresses moments of the operators $P_{n,m}(\cdot,t)$ in terms of more familiar moments of the Bernstein polynomials. After this we show that the sequence $\{P_{n,m}(\cdot,t)\}$ is L_p-bounded. Next we prove that

the sequence $\{P_{n,m}(\cdot,t)\}$ approximates continuous functions. This, together with the L_p-boundedness of the sequence $\{P_{n,m}(\cdot,t)\}$, shows that it is an L_p-approximating sequence.

Lemma 3.1.1. Let $k \in \mathbb{N}$. Then

(i) If $k \leq m$, $P_{n,m}((u-t)^k,t) = 0$, $t \in I$;

(ii) $P_{n,m}((u-t)^{m+1},t) = (-1)^m P_n(\prod_{i=0}^{m}(u-t+\frac{i}{n^{1/2}}),t)$

$$= (-1)^m \frac{(n+1)}{p(t)} \{\sum_{r=0}^{m} \frac{a_r}{n^{r/2}} B_{n+1}((u-t)^{m-r+3},t)\},$$

where a_r's are certain positive numbers;

(iii) If $k > m+1$,

$$P_{n,m}((u-t)^k,t) = \frac{(n+1)}{p(t)} \{\sum_{r=0}^{k-1} \frac{b_r}{n^{r/2}} B_{n+1}((u-t)^{k-r+2},t)\},$$

where b_r's are certain constants and $p(t) = t(1-t)$.

Proof. By (1.6.1) we have

(3.1.1) $$P_{n,m}((u-t)^k,t) =$$

$$\int_0^1 K(n,t,u)\{\sum_{j=0}^{m} \frac{n^{j/2}}{j!}(\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}}))(\Delta^j(u-t)^k)\}\,du,$$

where $\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}})$ for $j = 0$ is interpreted as 1 and

$$\Delta(u-t)^k = (\Delta(x-t)^k)_x\,(u)$$

$$= (u+\frac{1}{n^{1/2}}-t)^k - (u-t)^k.$$

For any choice of points t_i, $i = 0,1,\ldots,m$, with $\delta = t_{i+1}-t_i$, from (1.1.4) we have

$$(3.1.2)\quad \sum_{j=0}^{m} \left\{ \frac{(-1)^j}{j!\,\delta^j} \left(\prod_{i=0}^{j-1} t_i \right) \Delta_\delta^j t_o^k \right\} = \begin{cases} (-1)^m \left(\prod_{i=0}^{m} t_i \right), & k = m+1, \\ 0 & ,k < m+1. \end{cases}$$

Putting $t_o = u-t$, $\delta = n^{-1/2}$ in (3.1.2) and taking into account the fact that

$$\Delta_\delta t_o^k = (\Delta_\delta x^k)_x (t_o) = (\Delta_{n^{-1/2}} x^k)_x (u-t)$$

$$= (\Delta_{n^{-1/2}}(x-t)^k)_x(u) = (\Delta(x-t)^k)_x(u),$$

we obtain

$$(3.1.3)\quad \sum_{j=0}^{m} \left\{ \frac{n^{j/2}}{j!} \left(\prod_{i=0}^{j-1} \left(t-u - \frac{i}{n^{1/2}}\right)\right)(\Delta^j(u-t)^k) \right\}$$

$$= \begin{cases} (-1)^m \left(\prod_{i=0}^{m} \left(u - t + \frac{i}{n^{1/2}}\right)\right), & k = m+1 \\ 0 & , k < m+1. \end{cases}$$

Hence (i) follows from (3.1.1) and (3.1.3)

(ii) We have from (3.1.1) and (3.1.3)

$$P_{n,m}((u-t)^{m+1},t) = (-1)^m P_n\left(\prod_{j=0}^{m} \left(u-t + \frac{j}{n^{1/2}}\right),t\right)$$

$$= (-1)^m \left\{ \sum_{r=0}^{m} \frac{a_r'}{n^{r/2}} P_n((u-t)^{m+1-r},t) \right\}.$$

This, alongwith Lemma 1.7.2, completes the proof.

(iii) Finally, let $k > m+1$. We have the expansion

$$\Delta^j(u-t)^k = \sum_{r=0}^{k-j} \frac{c_r(u-t)^{k-j-r}}{n^{(j+r)/2}}$$

for certain constants c_r's. And

$$\prod_{i=0}^{j-1} \left(t-u - \frac{i}{n^{1/2}}\right) = \sum_{r_1=0}^{j-1} \frac{d_{r_1}(t-u)^{j-r_1}}{n^{r_1/2}}, \text{ say,}$$

where d_r's are certain constants. Then

$$\sum_{j=0}^{m} \left\{ \frac{n^{j/2}}{j!} \left(\prod_{i=0}^{j-1} \left(t-u-\frac{i}{n^{1/2}}\right)\right) \Delta^j (u-t)^k \right\}$$

$$= \sum_{j=0}^{m} \left\{ \frac{n^{j/2}}{j!} \left(\sum_{r_1=0}^{j-1} \frac{d_{r_1}(t-u)^{j-r_1}}{n^{r_1/2}} \right)\left(\sum_{r_2=0}^{k-j} \frac{c_{r_2}(u-t)^{k-j-r_2}}{n^{(j+r_2)/2}} \right)\right\}$$

$$= \sum_{r=0}^{k-1} \frac{b'_r}{n^{r/2}} (u-t)^{k-r},$$

where b_r's are certain constants. Hence, from (3.1.1) and Lemma 1.7.2 we obtain

$$P_{n,m}((u-t)^k,t) = \sum_{r=0}^{k-1} \frac{b'_r}{n^{r/2}} P_n((u-t)^{k-r},t)$$

$$= \frac{(n+1)}{p(t)} \left\{ \sum_{r=0}^{k-1} \frac{b_r}{n^{r/2}} B_{n+1}((u-t)^{k-r+2},t)\right\}$$

completing the proof of the lemma.

Corollary 3.1.2. Following holds for $t \in I$.

(3.1.4) $$P_{n,m}((u-t)^{m+1},t) = (-1)^m \frac{p_{m+1}(t)}{n^{(m+1)/2}} + o\left(\frac{1}{n^{(m+1)/2}}\right), (n \to \infty),$$

where $p_{m+1}(t)$ is a polynomial in t of degree $\leq m+1$ and $p_{m+1}(t) > 0$ for $t \in I^o$ (interior of I). The o-term holds uniformly with respect to $t \in I$.

Proof. By (ii) of Lemma 3.1.1

$$P_{n,m}((u-t)^{m+1},t) = (-1)^m \frac{(n+1)}{p(t)} \{ \sum_{r=0}^{m} \frac{a_r}{n^{r/2}} B_{n+1}((u-t)^{m-r+3},t) \}, a_r > 0,$$

$$(3.1.5) \qquad = (-1)^m \frac{(n+1)}{p(t)} \{ \sum_{r^1} + \sum_{r^2} \},$$

where $\sum_{r^1}$ and $\sum_{r^2}$ denote the summations for which $m-r+3$ is odd and even, respectively.

When $m-r+3$ is odd integer, by Lemma 1.7.1,

$(n+1)^{m-r+3}\ B_{n+1}((u-t)^{m-r+3},t)$ is a polynomial in $(n+1)$ of degree $\frac{m-r+2}{2}$ and in t of degree $\leq m-r+3$. Hence

$$\frac{n^{-r/2}}{p(t)}\ B_{n+1}((u-t)^{m-r+3},t) = O(\frac{1}{n^{(m+4)/2}}), \quad (n \to \infty).$$

This implies by (3.1.5) that

$$(3.1.6) \qquad (-1)^m \frac{(n+1)}{p(t)} (\sum_{r^1}) = o(\frac{1}{n^{(m+1)/2}}), \ (n \to \infty).$$

Again by Lemma 1.7.1 if $m-r+3$ is even integer we have

$$B_{n+1}((u-t)^{m-r+3},t) = \frac{(p(t))^{(m-r+3)/2}}{n^{(m-r+3)/2}} \cdot \frac{(m-r+3)!}{(\frac{m-r+3}{2})!} \cdot \frac{1}{2^{(m-r+3)/2}}$$

$$+ o(n^{-(m-r+3)/2}), \ (n \to \infty),$$

where the o-term holds uniformly in $t \in I$.

This implies, by (3.1.5) and taking into account positivity of a_r's, that

$$(3.1.7) \quad (-1)^m \frac{(n+1)}{p(t)} (\sum_{r^2}) = (-1)^m p_{m+1}(t) n^{-(m+1)/2}$$

$$+ o(n^{-(m+1)/2}), \ (n \to \infty).$$

Hence the proof follows from (3.1.5), (3.1.6) and (3.1.7).

Theorem 3.1.3. Let $1 \leq p < \infty$ and $f \in L_p(I)$. Then, for a fixed positive number ℓ, there holds for sufficiently large values of n

$$(3.1.8) \quad ||P_{n,m}(f,t)||_{L_p(I_2)} \leq M\{||f||_{L_p(I_1)} + n^{-\ell}||f||_{L_p(I)}\},$$

M being a constant.

Proof. We have $||P_{n,m}(f,t)||_{L_p(I_2)}$

$$= ||\int_0^1 K(n,t,u)\{\sum_{j=0}^{m} \frac{n^{j/2}}{j!}(\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}}))\Delta^j f(u)\}du||_{L_p(I_2)}$$

(where $\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}})$ for $j = 0$ is to be interpreted as 1)

$$\leq \sum_{j=0}^{m} \frac{n^{j/2}}{j!}||\int_0^1 K(n,t,u)(\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}}))\Delta^j f(u)du||_{L_p(I_2)}$$

$$(3.1.9) \qquad = \sum_{j=0}^{m} \frac{n^{j/2}}{j!} ||J_j||_{L_p(I_2)}, \text{ say.}$$

We choose numbers a^* and b^* satisfying $a_1 < a^* < a_2 < b_2 < b^* < b_1$. Writing $\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}})$ in terms of powers of $(t-u)$ we see from (3.1.9) that a typical term in J_j can be written as

$$\frac{c}{n^{r/2}}\int_0^1 K(n,t,u)(t-u)^{j-r}\Delta^j f(u)du = T_1(t), \text{ say,}$$

where $T_1(t) = T_1(t;j,r)$, c is a constant and $0 \leq r \leq j-1$.

Using Lemma 2.3.3 we obtain an L_p-bound for $T_1(t)$

$$||T_1||_{L_p(I_2)} \leq \frac{M_1}{n^{r/2}}\left\{\frac{1}{n^{(j-r)/2}}||\Delta^j f||_{L_p[a^*,b^*]} + \frac{1}{n^{\ell}}||\Delta^j f||_{L_p(I)}\right\}.$$

Hence for large values of n

$$||T_1||_{L_p(I_2)} \leq M_2\left\{\frac{1}{n^{j/2}}||f||_{L_p(I_1)} + \frac{1}{n^{\ell}}||f||_{L_p(I)}\right\}.$$

The theorem follows from above estimate of $T_1(t;j,r)$ and (3.1.9).

Theorem 3.1.4. Let $f(t)$ be continuous on I, Then

$$\lim_{n\to\infty} P_{n,m}(f,t) = f(t) \tag{3.1.10}$$

holds uniformly on I_1.

Proof. Since $P_n(1,t) = 1$, we have

$$P_{n,m}(f,t)-f(t) = \int_0^1 K(n,t,u)(f(u)-f(t))du$$

$$+ \sum_{j=1}^{m}\left\{\frac{n^{j/2}}{j!}\int_0^1 K(n,t,u)\left(\prod_{i=0}^{j-1}\left(t-u-\frac{i}{n^{1/2}}\right)\right)\Delta^j f(u)du\right\}$$

$$= J_0 + \sum_{j=1}^{m}\frac{n^{j/2}}{j!}J_j, \text{ say.} \tag{3.1.11}$$

Now $f \in C(I)$ implies that, given an arbitrary $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x)-f(y)| < \epsilon \text{ , whenever } |x-y| \leq \delta. \tag{3.1.12}$$

Using (3.1.12) and (1.7.5) it is easily seen that

$$|J_o| \leq \epsilon + 2\|f\|_{C(I)}\, \delta^{-2} \int_{|u-t|>\delta} K(n,t,u)(u-t)^2\, du$$

(3.1.13) $$\leq \epsilon + \frac{M_1}{n}, \text{ say.}$$

Now, a typical term in J_j is of the type

$$\frac{c}{n^{r/2}} \int_0^1 K(n,t,u)(t-u)^{j-r}\, \Delta^j f(u)\, du,$$

where $0 \leq r \leq j-1$ and $c = c(j,r)$ is a scalar.

This can be written as $\frac{c}{n^{r/2}} \int_0^1 K(n,t,u)(t-u)^{j-r}(\Delta^j f(u) - \Delta^j f(t))du$

(because $\Delta^j f(t) = 0$, as Δ acts on the u-part only)

$$= \frac{c}{n^{r/2}} \{ \sum_{s=0}^{j} \binom{j}{s}(-1)^{j-s} \{ \int_0^1 K(n,t,u)(t-u)^{j-r}(f(u+\frac{s}{n^{1/2}})-f(t))du \}\}$$

(3.1.14) $$= \frac{c}{n^{r/2}} \{ \sum_{s=0}^{j} \binom{j}{s}(-1)^{j-s}\, \Sigma_s \} \text{ say.}$$

We estimate the term Σ_s separately. We break the integration w.r.t. u into two parts corresponding to δ. Using (3.1.12) and then Corollary (1.7.6)

$$|\Sigma_s| \leq \int_{|u-t+\frac{s}{n^{1/2}}| \leq \delta} K(n,t,u)|t-u|^{j-r}|f(u+\frac{s}{n^{1/2}}) - f(t)|\, du$$

$$+ \int_{|u-t+\frac{s}{n^{1/2}}| > \delta} K(n,t,u)|t-u|^{j-r}|f(u+\frac{s}{n^{1/2}}) - f(t)|\, du$$

$$\leq \epsilon \int_0^1 K(n,t,u)|u-t|^{j-r}\, du$$

$$+ 2\|f\|_{C(I)}\, \delta^{-2} \int_0^1 K(n,t,u)|u-t|^{j-r}|u-t+\frac{s}{n^{1/2}}|^2\, du$$

$$\leq M_2\{\epsilon\, n^{\frac{r-j}{2}}$$

$$+\int_0^1 K(n,t,u)|u-t|^{j-r}(|u-t|^2+\frac{2s}{n^{1/2}}|u-t|+\frac{s^2}{n})du\}$$

$$\leq M_2'(\epsilon\, n^{\frac{r-j}{2}}+n^{\frac{r-j-2}{2}}). \tag{3.1.15}$$

Collecting (3.1.13), (3.1.14) and (3.1.15) we see that for each j

$$\left|\frac{n^{j/2}}{j!}\cdot J_j\right| \leq M(\epsilon+\frac{1}{n}) . \tag{3.1.16}$$

The estimates (3.1.16) and the arbitrariness of ϵ prove (3.1.10).

Corollary 3.1.5. Let $f(t)$ be continuous on I. Then

$$\lim_{n\to\infty} P_{n,m}(f,t) = f(t) \text{ uniformly on } I. \tag{3.1.17}$$

Proof. We extend f beyond $[0,1]$ such that the extended function is continuous throughout $\mathbb{R}$. This is done by defining function $\bar{f}(x)$ as follows.

$\bar{f}(x) = f(x),\ x \in I,$

$\bar{f}(x) = f(1),\ x > 1.$

Then proceeding as in the proof of Theorem 3.1.4 we obtain (3.1.17).

Corollary 3.1.6. Let $1 \leq p < \infty$ and $f \in L_p(I)$. Then there holds

$$||f(t)-P_{n,m}(f,t)||_{L_p(I_2)} = o(1), \quad (n\to\infty). \tag{3.1.18}$$

The proof follows from Theorems 3.1.3 and 3.1.4.

3.2 ERROR ESTIMATES AND A DIRECT THEOREM

In this section we obtain bounds for error in L_p-approximation by the sequence of operators $\{P_{n,m}(\cdot,t)\}$ and prove a Voronovskaja type asymptotic formula. The bounds are given in terms of L_p-norm of derivatives of the function and also in terms of (m+1)th integral modulus of smoothness of the function.

Theorem 3.2.1. Let $1 < p < \infty$ and $f \in L_p(I)$. If f has m+1 derivatives on I_1 with $f^{(m)} \in A.C.\ (I_1)$ and $f^{(m+1)} \in L_p(I_1)$, then for any fixed positive number ℓ and sufficiently large values of n

$$(3.2.1)\quad ||P_{n,m}(f,t)-f(t)||_{L_p(I_2)}$$

$$\leq M\{\frac{1}{n^{(m+1)/2}}||f^{(m+1)}||_{L_p(I_1)} + \frac{1}{n^{\ell}}||f||_{L_p(I)}\},$$

where M is a certain constant.

Proof. For $t \in I_2$ and $u \in I_1$ we can write

$$(3.2.2)\quad f(u) = \sum_{r=0}^{m} \frac{(u-t)^r}{r!} f^{(r)}(t) + \frac{1}{m!}\int_t^u (u-w)^m f^{(m+1)}(w)\,dw.$$

We choose numbers a^* and b^* such that $a_1 < a^* < a_2 < b_2 < b^* < b_1$. Let $\chi(u)$ be the characteristic function of $[a^*,b^*]$. Writing $F(t,u) = \sum_{j=0}^{m} \frac{n^{j/2}}{j!}\left(\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}})\right)\Delta^j f(u)$, where $\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}})$ for $j = 0$ is to be interpreted as 1.

We have

$$P_{n,m}(f,t)-f(t) = \int_0^1 K(n,t,u)(F(t,u)-f(t))\,du$$

$$= \int_0^1 \chi(u)K(n,t,u)(F(t,u)-f(t))\,du$$

$$+ \int_0^1 (1-\chi(u))K(n,t,u)(F(t,u)-f(t))\,du$$

$$(3.2.3) \qquad = J_1(t) + J_2(t), \text{ say.}$$

We first obtain a bound for $||J_1||_{L_p(I_2)}$. Using (3.2.2) we have for $t \in I_2$

$\chi(u)(F(t,u)-f(t)) =$

$$\chi(u)\Big\{ \sum_{r=0}^{m} \frac{f^{(r)}(t)}{r!} \Big\{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!}\Big(\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})\Big)(\Delta^j(u-t)^r)\Big\}-f(t)\Big\}$$

$$+\chi(u)\frac{1}{m!}\Big\{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!}\Big(\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})\Big)\Delta^j\Big(\int_t^u (u-w)^m f^{(m+1)}(w)dw\Big)\Big\} .$$

In view of (3.1.3) the above expression reduces to

$\chi(u)(F(t,u)-f(t))$

$$= \frac{1}{m!}\chi(u)\Big\{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!}\Big(\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})\Big)\Delta^j\Big(\int_t^u (u-w)^m f^{(m+1)}(w)dw\Big)\Big\} .$$

Hence

$$J_1(t) = \frac{1}{m!}\int_0^1 \chi(u)K(n,t,u)\Big\{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!}\Big(\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})\Big) \times$$

$$\times\ \Delta^j\Big(\int_t^u (u-w)^m f^{(m+1)}(w)dw\Big)\Big\}\,du .$$

Expanding $\Delta^j h(u)$ as $\sum_{s=0}^{j} \binom{j}{s}(-1)^{j-s}h(u+\frac{s}{n^{1/2}})$ and product $\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})$ as a finite sum in powers of $(t-u)$, a typical

component of $J_1(t)$ is of the type

$$c\int_0^1 \chi(u)K(n,t,u)n^{(j-r)/2}(t-u)^{j-r}\int_t^{u+\frac{s}{n^{1/2}}}(u-w+\frac{s}{n^{1/2}})^m f^{(m+1)}(w)dw\,du$$

$$= T_2(t), \text{ say,}$$

where $T_2(t) = T_2(t;j,r,s)$, c is a scalar and $j,r,s \in \mathbb{N}^0$ satisfy $0 \leq j \leq m$, $0 \leq r \leq j-1$, $0 \leq s \leq j$. This can be written as

$$c\,\{\sum_{k=0}^{m}\binom{m}{k}s^{m-k}\, n^{(j+k-m-r)/2}\int_0^1 \chi(u)\,K(n,t,u)(t-u)^{j-r}\,\times$$

$$\times\,\{\int_t^u (u-w)^k f^{(m+1)}(w)dw + \int_u^{u+\frac{s}{n^{1/2}}}(u-w)^k f^{(m+1)}(w)dw\,\}\,du\,\}$$

$$(3.2.4) \qquad = c\,\{\sum_{k=0}^{m}\binom{m}{k}\,s^{m-k}\,(T_{21}(t) + T_{22}(t))\}, \text{ say.}$$

It follows from the estimate of J_1 in Proposition 2.2.2 that

$$(3.2.5) \qquad ||T_{21}||_{L_p(I_2)} \leq \frac{M_1}{n^{(m+1)/2}}\,||f^{(m+1)}||_{L_p[a_2^*,b_2^*]}.$$

A bound for $||T_{22}||_{L_p(I_2)}$ is obtained as follows. We have

$$|T_{22}(t)| =$$

$$|\int_0^1 \chi(u)K(n,t,u)n^{(j+k-m-r)/2}(t-u)^{j-r}\,\{\int_u^{u+\frac{s}{n^{1/2}}}(u-w)^k f^{(m+1)}(w)dw\}du|$$

$$\leq s^k n^{(j-m-r)/2}\int_0^1 \chi(u)K(n,t,u)|t-u|^{j-r}\{\int_u^{u+\frac{s}{n^{1/2}}}|f^{(m+1)}(w)|dw\}du\ .$$

Applying Jensen's inequality twice we obtain

$$|T_{22}(t)|^p \leq$$

$$(s^k n^{(j-m-r)/2})^p \int_0^1 \chi(u)K(n,t,u)|t-u|^{(j-r)p} \times$$

$$\times \left\{ \int_u^{u+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|\,dw \right\}^p du$$

$$\leq (s^k n^{(j-m-r)/2})^p \int_0^1 \chi(u)K(n,t,u)|t-u|^{(j-r)p} \left(\frac{s}{n^{1/2}}\right)^{p-1} \times$$

$$(3.2.6) \qquad \times \int_u^{u+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|^p dw\,du$$

Using Fubini's theorem (to interchange integrals in u and t) and then applying Proposition 2.1.1 and Lemma 1.7.5 in the next step we obtain a bound for the following :

$$\int_{a_2}^{b_2} \int_0^1 K(n,t,u)\chi(u)|t-u|^{(j-r)p} \left(\int_u^{u+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|^p\, dw\right) du\, dt$$

$$= \int_0^1 \int_{a_2}^{b_2} K(n,t,u)\chi(u)|t-u|^{(j-r)p} \left(\int_u^{u+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|^p\, dw\right) dt\, du$$

$$= \int_0^1 \chi(u)\left(\int_{a_2}^{b_2} K(n,t,u)|t-u|^{(j-r)p}\, dt\right)\left(\int_u^{u+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|^p\, dw\right)du$$

$$\leq \frac{M_2}{n^{(j-r)p/2}} \int_0^1 \chi(u)\left(\int_u^{u+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|^p\, dw\right) du.$$

Let $\chi_u(w)$ denote the characteristic function of the interval $\left[u, u+\frac{s}{n^{1/2}}\right]$. Then, making use of Fubini's theorem the above expression is

$$M_2\, n^{(r-j)p/2} \int_0^1 \chi(u)\Big(\int_u^{u+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|^p\, \chi_u(w)\, dw\Big)\, du$$

$$\leq M_2\, n^{(r-j)p/2} \int_0^1 \chi(u)\Big(\int_{a^*}^{b^*+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|^p \chi_u(w) dw\Big) du$$

$$= M_2\, n^{(r-j)p/2} \int_{a^*}^{b^*+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|^p \Big(\int_0^1 \chi(u)\chi_u(w) du\Big)\, dw$$

$$= M_2\, n^{(r-j)p/2} \int_{a^*}^{b^*+\frac{s}{n^{1/2}}} |f^{(m+1)}(w)|^p \Big\{ \int_{w-\frac{s}{n^{1/2}}}^{w} \chi(u)\, du \Big\}\, dw$$

$$\leq M_2 s\, n^{\frac{(r-j)p-1}{2}}\, ||f^{(m+1)}||^p_{L_p[a^*,b^*+\frac{s}{n^{1/2}}]}.$$

This implies by (3.2.6) that for large values of n

$$(3.2.7) \qquad ||T_{22}(t)||_{L_p(I_2)} \leq \frac{M_2'}{n^{(m+1)/2}} ||f^{(m+1)}||_{L_p(I_1)}.$$

Thus it follows from (3.2.4), (3.2.5) and (3.2.7) that

$$||T_2(t)||_{L_p(I_2)} \leq \frac{M_3}{n^{(m+1)/2}} ||f^{(m+1)}||_{L_p(I_1)},$$

and hence $||J_1||_{L_p(I_2)} \leq \frac{M_3'}{n^{(m+1)/2}} ||f^{(m+1)}||_{L_p(I_1)}$.

It remains to obtain an estimate of J_2. For this we write

$$J_2(t) = \int_0^1 (1-\chi(u))K(n,t,u)F(t,u)du - f(t)\int_0^1 (1-\chi(u))K(n,t,u)du$$

$$(3.2.8) \qquad = J_{21}(t) - J_{22}(t), \text{ say.}$$

The presence of the factor $(1-\chi(u))$ implies, by Corollary 1.7.7, that for all $t \in I_2$

$$\int_0^1 K(n,t,u)(1-\chi(u))\,du \leq \frac{M_4}{n^\ell},$$

and hence

$$(3.2.9) \qquad ||J_{22}||_{L_p(I_2)} \leq \frac{M_4}{n^\ell} ||f||_{L_p(I_2)}.$$

A general component function of $J_{21}(t)$ is of the type

$$c \int_0^1 (1-\chi(u))K(n,t,u)\, n^{(j-r)/2} (t-u)^{j-r} f(u+\frac{s}{n^{1/2}}) = T(t), \text{ say,}$$

where $T(t) = T(t;j,r,s)$, $0 \leq r \leq j-1$, $0 \leq s \leq j$, and c is a scalar (when $j=0$, $r=0$).

It follows from Lemma 2.3.3 that also

$$||T(t)||_{L_p(I_2)} \leq \frac{M_4'}{n^\ell} ||f||_{L_p(I)},$$

and hence

$$(3.2.10) \qquad ||J_{21}||_{L_p(I_2)} \leq \frac{M_5}{n^\ell} ||f||_{L_p(I)}.$$

Finally, we obtain from (3.2.8), (3.2.9) and (3.2.10)

$$||J_2||_{L_p(I_2)} \leq \frac{M_5'}{n^\ell} ||f||_{L_p(I)}.$$

The theorem now follows from (3.2.3) and the L_p estimates of J_1 and J_2.

Corollary 3.2.2. Let $1 < p < \infty$ and $f \in L_p(I)$. If f has $m+1$ derivatives with $f^{(m)} \in A.C.(I)$ and $f^{(m+1)} \in L_p(I)$, then

$$(3.2.11) \qquad ||P_{n,m}(f,t)-f(t)||_{L_p(I)} \leq M\, n^{-(m+1)/2}||f^{(m+1)}||_{L_p(I)},$$

where M is a constant.

Proof. We proceed as in the proof of Theorem 3.2.1 with the characteristic function of $[a^*,b^*]$ replaced by the characteristic function of I. To obtain a bound for $T_{21}(t)$ we utilise the second assertion of Proposition 2.2.2.

Theorem 3.2.3. Let $f \in L_1(I)$. If f has m derivatives over the set I_1 with $f^{(m-1)} \in A.C.(I_1)$ and $f^{(m)} \in B.V.(I_1)$ then for any fixed positive number ℓ and sufficiently large values of n

$$(3.2.12) \qquad ||P_{n,m}(f,t)-f(t)||_{L_1(I_2)}$$

$$\leq M\{\frac{1}{n^{(m+1)/2}}||f^{(m)}||_{B.V.(I_1)} + \frac{1}{n^{\ell}}||f||_{L_1(I)}\},$$

where M is a certain constant.

Proof. With the stated assumptions on f, it follows from Theorem 14.1 of Saks [61] that for all $u \in I_1$ and almost all $t \in I_2$

$$(3.2.13) \qquad f(u) = \sum_{i=0}^{m-1} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{(u-t)^m}{m!} f^{(m)}(t)$$

$$+ \frac{1}{m!}\int_t^u (u-w)^m\, df^{(m)}(w).$$

With $\chi(u)$ as the characteristic function of $[a^*,b^*]$ (where a^*,b^* are as before) we have

$$P_{n,m}(f,t)-f(t) = \int_0^1 K(n,t,u)\chi(u)(F(t,u)-f(t))\,du$$

$$+ \int_0^1 K(n,t,u)(1-\chi(u))(F(t,u)-f(t))\,du$$

$$= J_1(t) + J_2(t), \text{ say.} \tag{3.2.14}$$

From (3.2.14), (3.2.13) and (i) of Lemma 3.1.1 it follows that for almost all $t \in I_2$,

$$J_1(t) = \frac{1}{m!}\int_0^1 \chi(u)K(n,t,u)\{\sum_{j=0}^{m} \frac{n^{j/2}}{j!}\left(\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}})\right)\times$$

$$\times\{\Delta^j(\int_t^u (u-w)^m\,df^{(m)}(w))\}\}du. \tag{3.2.15}$$

A typical component of $J_1(t)$ after expanding Δ^j can be written as

$$c\, n^{(j-r)/2}\int_0^1 \chi(u)K(n,t,u)(t-u)^{j-r}\left(\int_t^{u+\frac{s}{n^{1/2}}}(u+\frac{s}{n^{1/2}}-w)^m df^{(m)}(w)\right)du$$

$$= T_3(t), \text{ say,}$$

where $T_3(t) = T_3(t;j,r,s)$, $0 \le j \le m$, $0 \le r \le j-1$, $0 \le s \le j$ and c is a scalar.

This is re-written as

$$T_3(t) = c\sum_{k=0}^{m}\binom{m}{k}s^{m-k}\{\int_0^1 K(n,t,u)\chi(u)\, n^{(j+k-r-m)/2}\,(t-u)^{j-r}\times$$

$$\times\{\int_t^u (u-w)^k df^{(m)}(w) + \int_u^{u+\frac{s}{n^{1/2}}}(u-w)^k\,df^{(m)}(w)\}du\}$$

$$(3.2.16) \quad = c\{\sum_{k=0}^{m} \binom{m}{k} s^{m-k} (T_{31}(t) + T_{32}(t))\} \text{ say.}$$

By Proposition 2.2.5

$$(3.2.17) \quad ||T_{31}(t)||_{L_1(I_2)} \leq \frac{M_1}{n^{(m+1)/2}} ||f^{(m)}||_{B.V.[a^*,b^*]} .$$

To obtain a L_p-bound for $T_{32}(t)$, we proceed as in the proof of the estimate of $T_{22}(t)$ of Theorem 3.2.1. Thus for large values of n

$$(3.2.18) \quad ||T_{32}(t)||_{L_1(I_2)} \leq \frac{M_1'}{n^{(m+1)/2}} ||f^{(m)}||_{B.V.(I_1)} .$$

It follows from (3.2.15) to (3.2.18) that

$$||J_1||_{L_1(I_2)} \leq \frac{M_2}{n^{(m+1)/2}} ||f^{(m)}||_{B.V.(I_1)} .$$

Also, as in the proof of Theorem 3.2.1, for any fixed positive number ℓ we get

$$||J_2||_{L_1(I_2)} \leq \frac{M_2}{n^{\ell}} ||f||_{L_1(I)},$$

for all n sufficiently large. These estimates of J_1 and J_2 and the fact that removing a set of measure zero does not affect the L_1-norm complete the proof.

Corollary 3.2.4. Let $f \in L_1(I)$. If f has m derivatives over I with $f^{(m-1)} \in A.C.(I)$ and $f^{(m)} \in B.V.(I)$, then there holds

$$(3.2.19) \quad ||P_{n,m}(f,t)-f(t)||_{L_1(I)} \leq \frac{M}{n^{(m+1)/2}} ||f^{(m)}||_{B.V.(I)},$$

where M is a constant.

To prove (3.2.19), we use the second assertion of Proposition 2.2.5 and proceed as in the proof of the above theorem.

Theorem 3.2.5. Let $1 \leq p < \infty$ and $f \in L_p(I)$. Then, for sufficiently large values of n,

(3.2.20)
$$||P_{n,m}(f,t)-f(t)||_{L_p(I_2)} \leq M\{\omega_{m+1}(f,n^{-1/2},p,I_1) + n^{-(m+1)/2}||f||_{L_p(I)}\},$$

where M is a constant.

Proof. With a^*,b^* as before, we have, for all sufficiently small values of η

$$||P_{n,m}(f,t)-f(t)||_{L_p(I_2)} \leq ||P_{n,m}(f-f_{\eta,m+1},t)||_{L_p(I_2)}$$

$$+ ||P_{n,m}(f_{\eta,m+1},t)-f_{\eta,m+1}(t)||_{L_p(I_2)}+||f_{\eta,m+1}-f||_{L_p(I_2)}.$$

By Theorem 3.1.3, taking $\ell = (m+1)/2$

(3.2.21)
$$||P_{n,m}(f-f_{\eta,m+1},t)||_{L_p(I_2)} \leq M_1\{||f-f_{\eta,m+1}||_{L_p[a^*,b^*]} + n^{-(m+1)/2}||f-f_{\eta,m+1}||_{L_p(I)}\}.$$

From Theorems 3.2.1, 3.2.3 and the fact

$$||f^{(m+1)}||_{L_1[a,b]} = ||f^{(m)}||_{B.V.[a,b]},$$

it follows that

$$(3.2.22)\quad ||P_{n,m}(f_{\eta,m+1},t)-f_{\eta,m+1}(t)||_{L_p(I_2)}$$

$$\leq M_2\{n^{-(m+1)/2}(||f^{(m+1)}_{\eta,m+1}||_{L_p[a^*,b^*]}+||f_{\eta,m}||_{L_p(I)})\}.$$

Thus (3.2.21) and (3.2.22) imply that

$$||P_{n,m}(f,t)-f(t)||_{L_p(I_2)} \leq M_3\ \{||f-f_{\eta,m+1}||_{L_p[a^*,b^*]}$$

$$+\ n^{-(m+1)/2}\{||f^{(m+1)}_{\eta,m+1}||_{L_p[a^*,b^*]}+||f||_{L_p(I)}+||f_{\eta,m+1}||_{L_p(I)}\}\}.$$

This, in conjunction with the estimates (1.3.2), (1.3.3) and (1.3.4), implies that for $\eta = n^{-1/2}$ and for sufficiently large values of n

$$||P_{n,m}(f,t)-f(t)||_{L_p(I_2)} \leq M\{\omega_{m+1}(f,n^{-1/2},p,I_1)$$

$$+\ n^{-(m+1)/2}\ ||f||_{L_p(I)}\}.$$

Theorem 3.2.6. Let $f \in C^{m+1}(I)$. Then

$$(3.2.23)\quad P_{n,m}(f,t)-f(t) = -\frac{(-1)^m\ p_{m+1}(t)}{(m+1)!\ n^{(m+1)/2}}\ f^{(m+1)}(t)$$

$$+\ o(\frac{1}{n^{(m+1)/2}}),\ (n \to \infty),$$

and

$$(3.2.24)\quad P_{n,m+1}(f,t)-f(t) = o(\frac{1}{n^{(m+1)/2}}),\quad (n \to \infty),$$

uniformly in $t \in I$, where $p_{m+1}(t)$ is as defined in Corollary 3.1.2.

Proof. For some ξ lying between u and t where $u,t \in I$,

$$f(u) = \sum_{i=0}^{m+1} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{(u-t)^{m+1}}{(m+1)!} (f^{(m+1)}(\xi)-f^{(m+1)}(t)).$$

Operating by $P_{n,m}(\cdot,t)$ on both sides of the above equation, it follows from (i) of Lemma 3.1.1 and Corollary 3.1.2 that

$$P_{n,m}(f,t)-f(t) = \frac{f^{(m+1)}(t)}{(m+1)!} P_{n,m}((u-t)^{m+1},t)$$

$$+ \frac{1}{(m+1)!} P_{n,m}((u-t)^{m+1}(f^{(m+1)}(\xi)-f^{(m+1)}(t)),t)$$

$$= \frac{(-1)^m}{(m+1)!} \frac{p_{m+1}(t)}{n^{(m+1)/2}} f^{(m+1)}(t)$$

$$(3.2.25) \qquad + \frac{1}{(m+1)!} P_{n,m}((u-t)^{m+1}(f^{(m+1)}(\xi)-f^{(m+1)}(t)),t)$$

$$+ o(\frac{1}{n^{(m+1)/2}}), \quad (n \to \infty).$$

Therefore, to complete the proof of (3.2.23) it remains to show that

$$(3.2.26) \quad P_{n,m}((u-t)^{m+1}(f^{(m+1)}(\xi)-f^{(m+1)}(t)),t) = o(\frac{1}{n^{(m+1)/2}}), \ (n \to \infty).$$

$$P_{n,m}((u-t)^{m+1}(f^{(m+1)}(\xi)-f^{(m+1)}(t)),\ t)$$

$$(3.2.27)\quad = \sum_{j=0}^{m} \Big\{ \frac{n^{j/2}}{j!} \int_0^1 K(n,t,u)\Big(\prod_{i=0}^{j-1}\big(t-u-\frac{i}{n^{1/2}}\big)\Big) \times$$

$$\times\ \Delta^{j}((u-t)^{m+1}(f^{(m+1)}(\xi)-f^{(m+1)}(t)))du\Big\}.$$

A typical component, of above to be estimated is

$$c\ n^{(j-r+k-m-1)/2}\int_0^1 K(n,t,u)(u-t)^{j-r+k}(f^{(m+1)}(\xi_s)-f^{(m+1)}(t))\ du$$

$$= T(t), \text{ say,}$$

where $0 \leq j \leq m$, $0 \leq r \leq j-1$, $0 \leq k \leq m+1$, c is a constant, and ξ_s lies between $u+sn^{-1/2}$ and t. $(0 \leq s \leq j)$.

As $f^{(m+1)} \in C(I)$, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f^{(m+1)}(x)-f^{(m+1)}(y)| < \epsilon, \text{ whenever } |x-y| < \delta.$$

This implies that

$$|(u-t)^{j-r+k}(f^{(m+1)}(\xi_s)-f^{(m+1)}(t))| \leq \epsilon|u-t|^{j-r+k}$$

$$+ \frac{2}{\delta^2}\,||f^{(m+1)}||_{C(I)}|u-t|^{j-r+k}|u-t+\frac{s}{n^{1/2}}|^2 .$$

The above inequality, in conjunction with Corollary 1.7.7, gives

$$|T(t)| \leq M_1\Big(\frac{\epsilon}{n^{(m+1)/2}} + \frac{1}{n^{(m+3)/2}}\Big).$$

From (3.2.27) since $\epsilon > 0$ is arbitrary, we see that

$$P_{n,m}((u-t)^{m+1}(f^{(m+1)}(\xi)-f^{(m+1)}(t)),t) = o\left(\frac{1}{n^{(m+1)/2}}\right), \quad (n \to \infty),$$

completing the proof of (3.2.26).

Since Lemma 3.1.1 implies that $P_{n,m+1}((u-t)^{m+1},t) = 0$, proceeding as in the proof of (3.2.23) we obtain (3.2.24).

3.3 INVERSE THEOREM

Theorem 3.2.5 of the last section is a direct estimate for the operators $\{P_{n,m}\}$. Here in Theorem 3.3.1 we prove a corresponding local inverse theorem over contracting intervals. The proof of the theorem is preceded by two lemmas which are needed in the main proof.

Theorem 3.3.1. Let $0 < \alpha < m+1$, $1 \leq p < \infty$ and $f \in L_p(I)$. Then

$$(3.3.1) \qquad ||P_{n,m}(f,t)-f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}), \quad (n \to \infty),$$

implies that

$$(3.3.2) \qquad \omega_{m+1}(f,\tau,p,I_2) = O(\tau^{\alpha}), \qquad (\tau \to 0).$$

Lemma 3.3.2. Let $j,k,s \in \mathbb{N}^0$, $1 \leq p < \infty$ and $h \in L_p(I)$. Then for a fixed positive number ℓ and all sufficiently large values of n

$$(3.3.3) \qquad ||(n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t)|\frac{\nu}{n}-t|^{j}\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |u-t|^{k} \times$$

$$\times \left|\int_{t}^{u+\frac{s}{n^{1/2}}} |h(w)|\,dw\right| du\}\}||_{L_p(I_2)}$$

$$\leq M\{n^{-(j+k+1)/2}\,||h||_{L_p(I_1)} + n^{-\ell}\,||h||_{L_p(I)}\},$$

where M is a certain constant.

Proof. We have

$$||(n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t)|\tfrac{\nu}{n}-t|^{j}\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}|u-t|^{k}\times$$

$$\times|\int_{t}^{u+\frac{s}{n^{1/2}}}|h(w)|dw|du\}||_{L_p(I_2)}$$

$$\leq ||(n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t)|\tfrac{\nu}{n}-t|^{j}\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}|u-t|^{k}\times$$

$$\times|\int_{t}^{u}|h(w)|dw|du\}||_{L_p(I_2)}$$

$$+ ||(n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t)|\tfrac{\nu}{n}-t|^{j}\{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}|u-t|^{k}\times$$

$$\times\int_{u}^{u+\frac{s}{n^{1/2}}}|h(w)|dw\,du\}||_{L_p(I_2)}$$

(3.3.4) $\qquad = J_1 + J_2$, say.

The estimate

$$J_1 \leq M_0\{n^{-(j+k+1)/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p(I)}\},$$

follows from Lemma 2.3.2.

To obtain a bound for J_2 we proceed as follows. Making repeated use of Jensen's inequality

$$J_2^p = \int_{a_2}^{b_2} \{ \sum_{\nu=0}^{n} p_{n\nu}(t)|\frac{\nu}{n} - t|^j \{\int_{\nu/(n+1)}^{(\nu+1)/(n+1)} (n+1)|u-t|^k \times$$

$$\times \{ \int_u^{u+\frac{s}{n^{1/2}}} |h(w)|dw\}du\}\}^p \, dt$$

$$\leq \sum_{\nu=0}^{n} \{ \int_{a_2}^{b_2} p_{n\nu}(t)|\frac{\nu}{n} - t|^{jp} \{ \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} (n+1)|u-t|^k \times$$

$$\times \{ \int_u^{u+\frac{s}{n^{1/2}}} |h(w)|dw\}du\}^p \, dt \}$$

$$\leq \sum_{\nu=0}^{n} \{ \int_{a_2}^{b_2} p_{n\nu}(t)|\frac{\nu}{n} - t|^{jp} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} (n+1)|u-t|^{kp} \times$$

$$\times \{ \int_u^{u+\frac{s}{n^{1/2}}} |h(w)|dw\}^p \, du \, dt \}$$

$$\leq \sum_{\nu=0}^{n} \{ \int_{a_2}^{b_2} p_{n\nu}(t)|\frac{\nu}{n} - t|^{jp} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} (n+1)|u-t|^{kp} \left(\frac{s}{n^{1/2}}\right)^{p-1} \times$$

$$\times \{ \int_u^{u+\frac{s}{n^{1/2}}} |h(w)|^p \, dw\}du \, dt .$$

Let $\chi(u)$ be the characteristic function of $[a^*,b^*]$, where $a_1 < a^* < a_2 < b_2 < b^* < b_1$.

Then $$J_2^p \leq (n+1)\left(\frac{s}{n^{1/2}}\right)^{p-1} \{ \sum_{\nu=0}^{n} \{ \int_{a_2}^{b_2} p_{n\nu}(t)|\frac{\nu}{n} - t|^{jp} \times$$

$$\times \{ \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} \chi(u)|u-t|^{kp} \{ \int_u^{u+\frac{s}{n^{1/2}}} |h(w)|^p \, dw\}du \quad +$$

$$+ \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} (1-\chi(u))|u-t|^{kp}\{ \int_u^{u+\frac{s}{n^{1/2}}} |h(w)|^p dw\}du\}dt\}\}$$

(3.3.5) $\qquad = J_{21} + J_{22}$, say.

Let $\delta = \min(a_2 - a^*, b^* - b_2)$. Then, by Corollary 1.7.7

$$J_{22} \le \frac{n M_1}{n^{(p-1)/2}} ||h||^p_{L_p(I)} \int_{a_2}^{b_2} \int_0^1 K(n,t,u)|u-t|^{2\ell p}\, du\, dt$$

$$(3.3.6) \qquad \le M_1' \, ||h||^p_{L_p(I)} (n^{-\ell p}.n).$$

Using Fubini's theorem to interchange integrals in u and t for J_{21}, we obtain

$$J_{21} = (n+1)\left(\frac{s}{n^{1/2}}\right)^{p-1} \{ \sum_{\nu=0}^{n} \{ \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} \chi(u) \int_u^{u+\frac{s}{n^{1/2}}} |h(w)|^p \times$$

$$\times \left(\int_{a_2}^{b_2} p_{n\nu}(t) \left|\frac{\nu}{n} - t\right|^{jp} |t-u|^{kp}\, dt\right) dw\, du \}\}.$$

We first obtain a bound for $\int_{a_2}^{b_2} p_{n\nu}(t)\left|\frac{\nu}{n} - t\right|^{jp} |t-u|^{kp}\, dt$, where

$u \in \left[\frac{\nu}{n+1}, \frac{\nu+1}{n+1}\right] \cap [a^*, b^*]$.

Proceeding as in the proof of Proposition 2.1.1 we obtain

$$\int_{a_2}^{b_2} p_{n\nu}(t)\left|\frac{\nu}{n} - t\right|^{jp} |t-u|^{kp}\, dt \le \frac{M_2}{n^{\left(\frac{j+k}{2}\right)p + 1}} .$$

Let $\chi_u(w)$ be the characteristic function of the interval $\left[u, u + \frac{s}{n^{1/2}}\right]$. Then

$$J_{21} \le \frac{M_2'}{n^{\left(\frac{j+k+1}{2}\right)p-1}} \{ \sum_{\nu=0}^{n} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} \chi(u) \left(\int_u^{u+\frac{s}{n^{1/2}}} |h(w)|^p dw\right) du \}$$

$$= \frac{M_2'}{n^{\left(\frac{j+k+1}{2}\right)p-1}} \int_0^1 \chi(u)\left(\int_u^{u+\frac{s}{n^{1/2}}} \chi_u(w)|h(w)|^p\,dw\right)du$$

$$\leq \frac{M_2'}{n^{\left(\frac{j+k+1}{2}\right)p-1}} \int_0^1 \chi(u)\left(\int_{a^*}^{b^*+\frac{s}{n^{1/2}}} \chi_u(w)|h(w)|^p\,dw\right)du \;.$$

Applying Fubini's theorem to the expression on the right side we have

$$J_{21} \leq \frac{M_2'}{n^{\left(\frac{j+k+1}{2}\right)p-1}} \int_{a^*}^{b^*+\frac{s}{n^{1/2}}} |h(w)|^p \left(\int_0^1 \chi_u(w)\,du\right)dw$$

$$= \frac{M_2'}{n^{\left(\frac{j+k+1}{2}\right)p-1}} \int_{a^*}^{b^*+\frac{s}{n^{1/2}}} |h(w)|^p \left(\int_{w-\frac{s}{n^{1/2}}}^{w} du\right)dw$$

(3.3.7) $$= \frac{M_2' s}{n^{\left(\frac{j+k+1}{2}\right)p}} ||h||^p_{L_p[a^*,b^*+\frac{s}{n^{1/2}}]} \;.$$

The lemma follows from (3.3.4), the estimate of J_1 and the inequalities (3.3.5) to (3.3.7).

Lemma 3.3.3. Let $1 \leq p < \infty$ and $h \in L_p(I)$ where supp $h \subset [a,b]$, $0 < a < b < 1$. Then

(3.3.8) $$||P^{(m+1)}_{n,m}(h,t)||_{L_p[a,b]} \leq Mn^{(m+1)/2}||h||_{L_p[a,b]} \;.$$

If, in addition, h has $m+1$ derivatives on $[a,b]$ with $h^{(m)} \in$ A.C. $[a,b]$ and $h^{(m+1)} \in L_p[a,b]$, then

(3.3.9) $$||P^{(m+1)}_{n,m}(h,t)||_{L_p[a,b]} \leq M_1\, ||h^{(m+1)}||_{L_p[a,b]} \;,$$

where M, M_1 are constants independent of n and h.

Proof. First we prove (3.3.8). We have

$$P_{n,m}^{(m+1)}(h,t) = \sum_{k=0}^{m+1} \binom{m+1}{k} \times$$

$$\times \int_0^1 K^{(m+1-k)}(n,t,u)\Big(\sum_{j=k}^{m} \frac{n^{j/2}}{j!}\Big(\prod_{i=0}^{j-1}\Big(t-u-\frac{i}{n^{1/2}}\Big)\Big) \Delta^j h(u)\Big)^{(k)} du$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k}\Big\{\int_0^1 K^{(m+1-k)}(n,t,u)\,\Big\{n^{k/2}\Delta^k h(u) + \sum_{j=k+1}^{m} \frac{n^{j/2}}{j!}\,\Delta^j h(u) \times$$

$$\times\Big\{\sum_{i_1=0}^{j-1}\ \sum_{\substack{i_2=0\\(i_2\neq i_1)}}^{j-1} \cdots\cdots \sum_{\substack{i_k=0\\(i_k\neq i_1,\ldots,i_{k-1})}}^{j-1}\ \prod_{i\neq i_1,\ldots,i_k}\Big(t-u-\frac{i}{n^{1/2}}\Big)\Big\}\Big\}du\Big\}$$

In view of Lemma 1.7.4 a typical component of $P_{n,m}^{(m+1)}(h,t)$ can be written as

$$c(n+1)\Big\{\sum_{i_1,j_1} n^{i_1+j_1}\, q_{i_1 j_1}^{(m+1-k)}(t)\Big\{\sum_{\nu=0}^{n} p_{n\nu}(t)\Big(\frac{\nu}{n}-t\Big)^{j_1} n^{r_1/2} \times$$

$$\times \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} (t-u)^{r_2}\, \Delta^{r_3} h(u)\, du\Big\}\Big\}T^{-(m+1-k)}$$

$$= T_4(t), \text{ say,}$$

where $i_1, j_1, r_1, r_2, r_3 \in \mathbb{N}^0$, c is a scalar and $2i_1+j_1 \le m+1-k$, $0 \le r_2 \le r_3-k$, $k \le r_1 \le r_3$, $k \le r_3 \le m$, $r_1-r_2 = k$, and $q_{i,j}^{(m)}(t)$ are as in Lemma 1.7.4. Since $T^{-(m+1-k)}$ are bounded on $[a,b]$, we have with $\theta = i_1+j_1+\frac{r_1}{2}$

$$|T_4(t)| \leq M_o \{ \sum_{i_1,j_1} n^{\theta} \{ \sum_{\nu=0}^{n} p_{n\nu}(t) \left|\frac{\nu}{n} - t\right|^{j_1} \times$$

$$\times (n+1) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |t-u|^{r_2} |\Delta^{r_3} h(u)| \, du \}\} .$$

Applying Jensen's inequality three times successively

$$|T_4(t)|^p \leq M_o' \{ \sum_{i_1,j_1} n^{\theta p} \{ \sum_{\nu=0}^{n} p_{n\nu}(t) \left|\frac{\nu}{n} - t\right|^{j_1} \times$$

$$\times (n+1) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |t-u|^{r_2} |\Delta^{r_3} h(u)| \, du \}^p \}$$

$$\leq M_o' \{ \sum_{i_1,j_1} n^{\theta p} \{ \sum_{\nu=0}^{n} p_{n\nu}(t) \left|\frac{\nu}{n} - t\right|^{j_1 p} \times$$

$$\times \{ (n+1) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |t-u|^{r_2} |\Delta^{r_3} h(u)| \, du\}^p \}\}$$

$$\leq M_o' \{ \sum_{i_1,j_1} n^{\theta p} \{ \sum_{\nu=0}^{n} p_{n\nu}(t) \left|\frac{\nu}{n} - t\right|^{j_1 p} \times$$

$$\times (n+1) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |t-u|^{r_2 p} |\Delta^{r_3} h(u)|^p \, du \}\} .$$

By Fubini's theorem

$$\int_{a_2}^{b_2} |T_4(t)|^p \, dt \leq M_o' \{ \sum_{i_1,j_1} n^{\theta p+1} \{ \sum_{\nu=0}^{n} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |\Delta^{r_3} h(u)|^p \times$$

$$\times \{ \int_{a_2}^{b_2} p_{n\nu}(t) \left|\frac{\nu}{n} - t\right|^{j_1 p} |t-u|^{r_2 p} \, dt\} du \}\} .$$

Proceeding as in the proof of Proposition 2.1.1 we obtain

$$\int_{a_2}^{b_2} |T_4(t)|^p \, dt \leq M_2 \{ \sum_{i_1, j_1} \frac{n^{\theta p+1}}{n^{1+(j_1+r_2)p/2}} \times$$

$$\times \{ \sum_{\nu=0}^{n} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} |\Delta^{r_3} h(u)|^p \, du \}\}$$

$$\leq M_3 \, n^{(m+1)p/2} \, ||h||^p_{L_p[a,b]} .$$

The last inequality is obtained by using the conditions on i_1, j_1, r_1 and r_2. This proves (3.3.8) because $T_4(t)$ is any typical component of $P^{(m+1)}_{n,m}(h,t)$.

To prove (3.3.9) we write

$$h(u) = \sum_{i=0}^{m} \frac{(u-t)^i}{i!} h^{(i)}(t) + \frac{1}{m!} \int_t^u (u-w)^m \, h^{(m+1)}(w) \, dw.$$

Since Lemmas 1.7.1 and 3.1.1 imply that $P_{n,m}(\cdot,t)$ maps algebraic polynomials into algebraic polynomials of same degree, we obtain

$$P^{(m+1)}_{n,m}(h,t) = \frac{1}{m!} P^{(m+1)}_{n,m} \left(\int_t^u (u-w)^m \, h^{(m+1)}(w) \, dw, \; t \right)$$

$$= \frac{1}{m!} \{ \sum_{k=0}^{m} \binom{m+1}{k} \int_0^1 K^{(m+1-k)}(n,t,u) \{ n^{k/2} \Delta^k \left(\int_t^u (u-w)^m h^{(m+1)}(w) dw\right)$$

$$+ \sum_{j=k+1}^{m} \frac{n^{j/2}}{j!} \Delta^j \left(\int_t^u (u-w)^m \, h^{(m+1)}(w) dw\right) \times$$

$$\times \{ \sum_{i_1=0}^{j-1} \sum_{\substack{i_2=0 \\ (i_2 \neq i_1)}}^{j-1} \cdots \sum_{\substack{i_k=0 \\ (i_k \neq i_1, \ldots, i_{k-1})}}^{j-1} \prod_{i \neq i_1, \ldots, i_k} \left(t-u- \frac{i}{n^{1/2}}\right)\}\} du \}$$

As before a typical component of the above expression is represented by

$$c(n+1)\ \{ \sum_{i_1,j_1} n^{i_1+j_1} q_{i_1 j_1}^{(m+1-k)}(t)\ \{ \sum_{\nu=0}^{n} p_{n\nu}(t)(\tfrac{\nu}{n}-t)^{j_1}\ n^{\frac{r_1+r_3-m}{2}} \times$$

$$\times \{ \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} (t-u)^{r_2} \{ \int_t^{u+\frac{r_4}{n^{1/2}}} (u-w)^{r_3}\ h^{(m+1)}(w)dw\}du\}\}\} T^{-(m+1-k)}$$

$$= T_5(t), \text{ say,}$$

where $i_1, j_1, r_1, r_2, r_3, r_4 \in \mathbb{N}^0$ satisfy $2i_1+j_1 \leq m+1-k, k \leq r_1 \leq j$, $0 \leq r_2 \leq j-k$, $0 \leq r_3 \leq m$, $0 \leq r_4 \leq m$, $r_1-r_2 = k$, and c is a scalar.

Proceeding in the manner of the proof of Lemma 3.3.2 one obtains

$$||T_5(t)||_{L_p[a,b]}$$

$$\leq M_5 ||h^{(m+1)}||_{L_p[a,b]} \{ \sum_{i_1,j_1} \{ n^{i_1+j_1} \times$$

$$\times\ n^{(r_1+r_3-m)/2} \cdot \frac{1}{n^{(j_1+r_2+r_3+1)/2}} \}\}$$

$$\leq M_5' ||h^{(m+1)}||_{L_p[a,b]} .$$

As $T_5(t)$ is any typical component of $P_{n,m}^{(m+1)}(h,t)$, we obtain (3.3.9).

Proof of Theorem 3.3.1. We choose pairs of points (x_i, y_i), $i = 1,2,3,4$ such that $a_1 < x_i < x_{i+1} < a_2 < b_2 < y_{i+1} < y_i < b_1$,

and a function $g \in C_0^{m+1}$ such that $\operatorname{supp} g \subset (x_3,y_3)$ and $g(t) = 1$ for $t \in [x_4,y_4]$.

Writing $fg = \bar{f}$, as in the proof of Theorem 2.3.1, for all values of $\gamma \leq \tau$ (τ being sufficiently small), we have

$$||\Delta_\gamma^{m+1} \bar{f}(t)||_{L_p[x_3,y_3]} \leq ||\Delta_\gamma^{m+1}(\bar{f}(t)-P_{n,m}(\bar{f},t))||_{L_p[x_3,y_3]}$$

$$+ \gamma^{m+1}\{||P_{n,m}^{(m+1)}(\bar{f}-\bar{f}_{\eta,m+1},t)||_{L_p[x_3,y_3]}$$

$$+ ||P_{n,m}^{(m+1)}(\bar{f}_{\eta,m+1},t)||_{L_p[x_3,y_3]}\},$$

where $\bar{f}_{\eta,m+1}$ is Steklov mean of $(m+1)$th order corresponding to $\bar{f}$. This, in conjunction with Lemma 3.3.3, for sufficiently small values of η, gives

$$||\Delta_\gamma^{m+1} \bar{f}(t)||_{L_p[x_3,y_3]} \leq ||\Delta_\gamma^{m+1}(\bar{f}(t)-P_{n,m}(\bar{f},t))||_{L_p[x_3,y_3]}$$

$$+ \gamma^{m+1}M_2\{n^{(m+1)/2}||\bar{f}-\bar{f}_{\eta,m+1}||_{L_p[x_3,y_3]}+||f_{\eta,m+1}^{(m+1)}||_{L_p[x_3,y_3]}\}.$$

Applying estimates (1.3.3) and (1.3.2), respectively for the second and the third terms of the right hand side of the above inequality, we obtain

$$(3.3.10)\quad ||\Delta_\gamma^{m+1} \bar{f}(t)||_{L_p[x_3,y_3]}$$

$$\leq ||\Delta_\gamma^{m+1}(\bar{f}(t)-P_{n,m}(\bar{f},t))||_{L_p[x_3,y_3]} +$$

$$+ M_2' \gamma^{m+1}(n^{(m+1)/2} + n^{-(m+1)})\, \omega_{m+1}(\bar{f}, n, p, [x_3, y_3]).$$

Now we show, by an induction on α, that

$$(3.3.11) \qquad ||\Delta_\gamma^{m+1}(\bar{f}(t) - P_{n,m}(\bar{f}, t))||_{L_p[x_3,y_3]} = O(n^{-\alpha/2}), (n \to \infty).$$

Having proved (3.3.11) we obtain (3.3.2) as in Theorem 2.3.1. First, assuming that $\alpha \leq 1$, by (3.3.1) for some ξ lying between u and t, we have

$$||P_{n,m}(fg, t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq ||P_{n,m}((f(u)-f(t))g(t), t)||_{L_p[x_3,y_3]}$$

$$+ ||P_{n,m}(f(u)(g(u)-g(t)), t)||_{L_p[x_3,y_3]}$$

$$\leq \frac{M_3}{n^{\alpha/2}} + ||P_{n,m}(f(u)(u-t)g'(\xi), t)||_{L_p[x_3,y_3]}$$

$$(3.3.12) \qquad = \frac{M_3}{n^{\alpha/2}} + J \text{ , say.}$$

A typical component of $P_{n,m}(f(u)(u-t)g'(\xi), t)$ can be written as

$$c\, n^{(j-r-i)/2} \int_0^1 K(n,t,u)(t-u)^{j-r-i+1} f(u + \frac{k}{n^{1/2}})g'(\xi_k)\, du$$

$$= T(t), \text{ say,}$$

where $i, j, k, r \in \mathbb{N}^0$, $0 \leq j \leq m$, $0 \leq k \leq j$, $0 \leq r \leq j-1$, $i = 0, 1$, ξ_k lies between $u + \frac{k}{n^{1/2}}$ and t and c is a scalar.

By Lemma 2.3.3 for large values of n and any fixed positive number ℓ

$$(3.3.13)\quad ||T(t)||_{L_p[x_3,y_3]} \le M_3' \{n^{-1/2}||f||_{L_p[x_2,y_2]} + n^{-\ell}\,||f||_{L_p(I)}\}.$$

Thus $J = O(n^{-1/2})$, $(n \to \infty)$.

This implies by (3.3.12) that

$$||P_{n,m}(fg,t)-(fg)(t)||_{L_p[x_3,y_3]} = O(n^{-1/2}), \quad (n \to \infty),$$

proving (3.3.11).

Next, we assume that for some $r \le m$, the theorem holds for all values of α satisfying $r-1 \le \alpha < r$. We are then to show that the theorem also remains valid for all α satisfying $r \le \alpha < r+1$.

With $f_{\eta,m+1}$ as the Steklov mean of (m+1)th order corresponding to f

$$||P_{n,m}(fg,t)-(fg)(t)||_{L_p[x_3,y_3]}$$

$$\le ||P_{n,m}((f(u)-f(t))g(t),t)||_{L_p[x_3,y_3]}$$

$$+ ||P_{n,m}(f(u)(g(u)-g(t)),t)||_{L_p[x_3,y_3]}$$

$$(3.3.14)\quad \le \frac{M_3}{n^{\alpha/2}} + ||P_{n,m}(f(u)(g(u)-g(t)),t)||_{L_p[x_3,y_3]}.$$

Now

$$||P_{n,m}(f(u)(g(u)-g(t)),t)||_{L_p[x_3,y_3]}$$

$$\leq ||P_{n,m}((f(u)-f_{n,m+1}(u))(g(u)-g(t)),t)||_{L_p[x_3,y_3]}$$

$$+||P_{n,m}((f_{n,m+1}(u)-f_{n,m+1}(t))(g(u)-g(t)),t)||_{L_p[x_3,y_3]}$$

$$+||P_{n,m}(f_{n,m+1}(t)(g(u)-g(t)),t)||_{L_p[x_3,y_3]}$$

(3.3.15) $= J_1 + J_2 + J_3$, say.

By Theorem 3.2.6 and (1.3.4) of Lemma 1.3.1

(3.3.16) $$J_3 \leq \frac{M_4}{n^{(m+1)/2}}.$$

Proceeding as in the estimate of J in (3.3.12) we obtain

$$J_1 \leq M_4' \{n^{-1/2}||f-f_{n,m+1}||_{L_p[x_2,y_2]}+n^{-\ell}||f-f_{n,m+1}||_{L_p(I)}\},$$

where ℓ is an arbitrarily fixed positive number.

Applying the estimates (1.3.3) and (1.3.4) to this inequality we obtain

(3.3.17) $$J_1 \leq M_5 \{n^{-1/2}\omega_{m+1}(f,n,p,[x_1,y_1])+n^{-\ell}||f||_{L_p(I)}\}.$$

A bound for J_2 is obtained as follows. For some ξ lying between u and t

$$(f_{n,m+1}(u)-f_{n,m+1}(t))(g(u)-g(t))$$

$$=\{\sum_{i=1}^{m}\frac{(u-t)^i}{i!}f^{(i)}_{n,m+1}(t)+\frac{1}{m!}\int_t^u(u-w)^m f^{(m+1)}_{n,m+1}(w)\,dw\}\times$$

$$\times\{\sum_{i=1}^{m-1}\frac{(u-t)^i}{i!}g^{(i)}(t)+\frac{(u-t)^m}{m!}g^{(m)}(\xi)\}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{m-1}\{\frac{1}{i!\,j!}f^{(i)}_{n,m+1}(t)g^{(j)}(t)(u-t)^{i+j}\}$$

$$+\frac{g^{(m)}(\xi)}{m!}\{\sum_{i=1}^{m}f^{(i)}_{n,m+1}(t)(u-t)^{i+m}\}$$

$$+\frac{1}{m!}\{\sum_{i=1}^{m-1}(\frac{g^{(i)}(t)}{i!}(u-t)^i\int_t^u(u-w)^m f^{(m+1)}_{n,m+1}(w)dw)\}$$

$$+\frac{1}{(m!)^2}\{g^{(m)}(\xi)(u-t)^m\int_t^u(u-w)^m f^{(m+1)}_{n,m+1}(w)\,dw\}.$$

Hence we obtain from above expansion and (3.3.15)

$$J_2\leq$$

$$\{\sum_{i=1}^{m}\sum_{j=1}^{m-1}\frac{1}{i!\,j!}\,||f^{(i)}_{n,m+1}(t)g^{(j)}(t)P_{n,m}((u-t)^{i+j},t)||_{L_p[x_3,y_3]}\}$$

$$+\frac{1}{m!}\{\sum_{i=1}^{m}||f^{(i)}_{n,m+1}(t)P_{n,m}((u-t)^{i+m}g^{(m)}(\xi),t)||_{L_p[x_3,y_3]}\}$$

$$+\frac{1}{m!}\{\sum_{i=1}^{m-1}\frac{1}{i!}\times$$

$$\times||g^{(i)}(t)\{P_{n,m}((u-t)^i(\int_t^u(u-w)^m f^{(m+1)}_{n,m+1}(w)dw),t)\}||_{L_p[x_3,y_3]}\}$$

$$+\frac{1}{(m!)^2}||P_{n,m}(g^{(m)}(\xi)(u-t)^m(\int_t^u(u-w)^m f^{(m+1)}_{n,m+1}(w)dw),t)||_{L_p[x_3,y_3]}$$

(3.3.18) $\qquad = J_{21} + J_{22} + J_{23} + J_{24}$, say.

We first obtain a bound for J_{23} and J_{24}. A typical component of

$$P_{n,m}(g^{(m)}(\xi)(u-t)^k(\int_t^u (u-w)^m f_{n,m+1}^{(m+1)}(w)dw),t)$$

$$= \int_0^1 K(n,t,u)\ \{\sum_{j=0}^{m} \frac{n^{j/2}}{j!}\ (\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}}))\ \times$$

$$\times\ \Delta^j(g^{(m)}(\xi)(u-t)^k(\int_t^u (u-w)^m\ f_{n,m+1}^{(m+1)}(w)dw))\}\ du\ ,$$

after expanding Δ^j and $\prod_{i=0}^{j-1}$ can be written as

$$c\ n^{(\theta-k)/2} \int_0^1 K(n,t,u)(u-t)^\theta\ g^{(m)}(\xi_{r_2})\ \times$$

$$\times\ \{\int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m\ f_{n,m+1}^{(m+1)}(w)\ dw\}\ du$$

(3.3.19) $\qquad = T(t)$, say,

where $T(t) = T(t;j,r_1,r_2,r_3)$, $\theta = j + r_3-r_1$, $0 \leq j \leq m$, $0 \leq r_1 \leq j-1$, $0 \leq r_2 \leq j$, $0 \leq r_3 \leq k$, ξ_{r_2} lies between $u + \frac{r_2}{n^{1/2}}$ and t and c is a scalar.

Let $\chi(u)$ be the characteristic function of $[c,d]$ where $x_2 < c < d < y_2$. Then

$$T(t) = c\ n^{(\theta-k)/2}\{\int_0^1 \chi(u)K(n,t,u)(u-t)^\theta g^{(m)}(\xi_{r_2})\ \times$$

$$\times\ \{\int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m\ f_{n,m+1}^{(m+1)}(w)\ dw\}du\ +$$

$$+ \int_0^1 (1-\chi(u))K(n,t,u)(u-t)^{\theta} g^{(m)}(\xi_{r_2})$$

$$\times \{ \int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m f_{n,m+1}^{(m+1)}(w)dw\}du\}$$

(3.3.20) $$= T_6(t) + T_7(t), \text{ say.}$$

It follows from the estimate of $T_2(t)$ in Theorem 3.2.1 that for sufficiently large values of n

$$||T_6(t)||_{L_p[x_3,y_3]} \leq \frac{M_5'}{n^{(k+m+1)/2}} ||f_{n,m+1}^{(m+1)}||_{L_p[x_2,y_2]} .$$

We have from (3.3.20)

$$T_7(t)| \leq M_6 ||f_{n,m+1}^{(m+1)}||_{L_1(I)} \{ \int_0^1 (1-\chi(u))K(n,t,u)|u-t|^{\theta} du\} n^{\theta/2}$$

The presence of the factor $(1-\chi(u))$ implies, by (1.7.6), that

$$|T_7(t)| \leq \frac{M_6'}{n^{\ell}} ||f_{n,m+1}^{(m+1)}||_{L_p(I)} \quad \text{for all } t \in [x_3,y_3] .$$

Consequently

$$||T_7(t)||_{L_p[x_3,y_3]} \leq \frac{M_7}{n^{\ell}} ||f_{n,m+1}^{(m+1)}||_{L_p(I)} .$$

The L_p-bounds for functions $T_6(t)$ and $T_7(t)$ give, by (3.3.20),

$$||T(t)||_{L_p[x_3,y_3]} \leq M_7' \{ \frac{1}{n^{(k+m+1)/2}} ||f_{n,m+1}^{(m+1)}||_{L_p[x_2,y_2]}$$

(3.3.21) $$+ \frac{1}{n^{\ell}} ||f_{n,m+1}^{(m+1)}||_{L_p(I)} \} .$$

Applying estimates (1.3.2) and (1.3.5) we obtain from (3.3.19) and (3.3.21) that

$$||P_{n,m}(g^{(m)}(\xi)(u-t)^k \int_t^u (u-w)^m f_{\eta,m+1}^{(m+1)}(w)\, dw,\ t)||_{L_p[x_3,y_3]}$$

$$(3.3.22)\quad \le M_8 \{ \frac{1}{n^{(k+m+1)/2}\ \eta^{m+1}}\ \omega_{m+1}(f,\eta,p,[x_1,y_1])$$

$$+ \frac{1}{n^{\ell}\ \eta^{m+1}} ||f||_{L_p(I)} \}.$$

The bounds for J_{21} and J_{22} follow as particular cases of (3.3.22). Thus

$$(3.3.23)\quad J_{23} \le M_8' \{ (\sum_{i=1}^{m-1} \frac{1}{n^{(i+m+1)/2}})\ \frac{1}{\eta^{m+1}}\ \omega_{m+1}(f,\eta,p,[x_1,y_1])$$

$$+ \frac{1}{n^{\ell}\ \eta^{m+1}} ||f||_{L_p(I)} \},$$

and

$$(3.3.24)\quad J_{24} \le M_8' \{ \frac{1}{n^{(2m+1)/2}\ \eta^{m+1}}\ \omega_{m+1}(f,\eta,p,[x_1,y_1])$$

$$+ \frac{1}{n^{\ell}\ \eta^{m+1}} ||f||_{L_p(I)} \}.$$

By Lemma 3.1.1 and Corollary 1.7.6

$$(3.3.25)\quad J_{21} \le M_9 \{ \sum_{i=1}^{m} \sum_{j=1}^{m-1} \frac{1}{n^{(i+j)/2}} ||f_{\eta,m+1}^{(i)}||_{L_p[x_3,y_3]} \},$$

where summation is taken only over those i,j which satisfy $i+j > m$.

This, in conjunction with Lemma 1.2.2, gives

$$J_{21} \leq \frac{M_9'}{n^{(m+1)/2}}(||f_{\eta,m+1}^{(m)}||_{L_p[x_3,y_3]} + ||f_{\eta,m+1}||_{L_p[x_3,y_3]}).$$

Applying estimates (1.3.2) and (1.3.4)

$$(3.3.26) \quad J_{21} \leq \frac{M_{10}}{n^{(m+1)/2}}(\frac{1}{\eta^m}\omega_m(f,\eta,p,[x_2,y_2]) + ||f||_{L_p(I)}).$$

A typical term in $P_{n,m}(g^{(m)}(\xi)(u-t)^{m+i},t)$ is represented by

$$c\, n^{(j+k-r-m-i)/2} \int_0^1 K(n,t,u)g^{(m)}(\xi_s)(u-t)^{j+k-r}\, du$$

$$= T_8(t), \text{ say,}$$

where $0 \leq j \leq m$, $0 \leq r \leq j-1$, $0 \leq s \leq j$, $0 \leq k \leq m+i$, ξ_s lies between $u + \frac{s}{n^{1/2}}$ and t, and c is a scalar.

Applying Corollary 1.7.7 we obtain

$$|T_8(t)| \leq \frac{M_{10}'}{n^{(m+i)/2}} \quad \text{for all } t \in [x_3,y_3].$$

Consequently

$$J_{22} \leq M_{11}(\sum_{i=1}^{m} \frac{1}{n^{(m+i)/2}} ||f_{\eta,m+1}^{(i)}||_{L_p[x_3,y_3]}).$$

As before, using Lemma 1.2.2 and the estimates (1.3.2) and (1.3.4) we get

$$(3.3.27) \quad J_{22} \leq \frac{M_{11}'}{n^{(m+1)/2}}(\frac{1}{\eta^m}\omega_m(f,\eta,p,[x_2,y_2]) + ||f||_{L_p(I)}).$$

Collecting (3.3.18), (3.3.23), (3.3.24), (3.3.26) and (3.3.27) we obtain

$$(3.3.28)\quad J_2 \le M_{12}\{\frac{1}{n^{(m+2)/2}\eta^{m+1}}\,\omega_{m+1}(f,\eta,p,[x_1,y_1])$$

$$+\frac{1}{n^{(m+1)/2}\eta^{m}}\,\omega_m(f,\eta,p,[x_2,y_2])\}$$

$$+(\frac{1}{n^{\ell}\eta^{m+1}}+\frac{1}{n^{(m+1)/2}})||f||_{L_p(I)}\}.$$

We have by the induction hypothesis

$$\omega_{m+1}(f,\eta,p,[x_1,y_1]) = O(\eta^{\alpha-1}),\quad (\eta \to 0).$$

This implies by Corollary 1.3.4 that

$$\omega_m(f,\eta,p,[x_1,y_1]) = O(\eta^{\alpha-1}),\quad (\eta \to 0).$$

Applying the estimates of mth and (m+1)th modulus of smoothness to (3.3.28) and taking $\eta = n^{-1/2}$, $\ell = m+1$

$$(3.3.29)\qquad J_2 \le \frac{M_{13}}{n^{\alpha/2}}.$$

Also by (3.3.17)

$$(3.3.30)\qquad J_1 \le \frac{M_{14}}{n^{\alpha/2}}.$$

Finally we obtain from (3.3.14), (3.3.15), (3.3.16), (3.3.29) and (3.3.30) that

$$||P_{n,m}(fg,t)-(fg)(t)||_{L_p[x_3,y_3]} = O(n^{-\alpha/2}),\ (n \to \infty).$$

This proves (3.3.11) and hence the proof of the theorem.

3.4 SATURATION THEOREM

The asymptotic formula for the operators $P_{n,m}(\cdot,t)$ (Theorem 3.2.6) gives an indication of the saturation behaviour of the operators. It is shown here that the operators $P_{n,m}(\cdot,t)$ are indeed saturated with the order $O(n^{-(m+1)/2})$. We get different saturation classes depending on whether $p = 1$ or $p > 1$. The trivial class consists of functions which are locally polynomials of degree m.

Theorem 3.4.1. Let $1 \leq p < \infty$ and $f \in L_p(I)$. Then, in the following statements, the implications "(i) ==> (ii) ==> (iii)" and "(iv) ==> (v) ==> (vi)" hold.

(i) $\|P_{n,m}(f,t)-f(t)\|_{L_p(I_1)} = O(n^{-(m+1)/2}), \ (n \to \infty)$;

(ii) f coincides a.e. on I_2 with a function F having m+1 derivatives such that (a) when $p > 1$, $F^{(m)} \in A.C.(I_2)$ and $F^{(m+1)} \in L_p(I_2)$, (b) when $p = 1$, $F^{(m-1)} \in A.C.(I_2)$ and $F^{(m)} \in B.V.(I_2)$;

(iii) $\|P_{n,m}(f,t)-f(t)\|_{L_p(I_3)} = O(n^{-(m+1)/2}), \ (n \to \infty)$;

(iv) $\|P_{n,m}(f,t)-f(t)\|_{L_p(I_1)} = o(n^{-(m+1)/2}), \ (n \to \infty)$;

(v) f coincides a.e. on I_2 with a polynomial of degree m;

(vi) $\|P_{n,m}(f,t)-f(t)\|_{L_p(I_3)} = o(n^{-(m+1)/2}), \ (n \to \infty)$.

Note. The implication "(ii) ==> (iii)" follows from Theorems 3.2.1 and 3.2.3, respectively for the cases $1 < p < \infty$ and $p = 1$. And "(v) ==> (vi)" follows from Theorem 3.2.6.

We shall first prove the following lemma.

Lemma 3.4.2. Let $h \in L_p(I)$, $1 \leq p < \infty$, have a compact support $\subset (0,1)$. Further, let h have m derivatives over I where $h^{(m-1)} \in A.C.(I)$ and $h^{(m)} \in L_p(I)$ and satisfies for all values of $\beta \in (0,1)$ the condition :

$$\omega(h^{(m)},\tau,p,I) \leq M\,\tau^{\beta}\ , \quad (\tau \to 0).$$

Then, for each $g \in C_0^{m+1}$ with supp $g \subset (0,1)$

$$(3.4.1) \quad |< P_{n,m}(h,t)-h(t),g(t) >|$$

$$\leq \frac{K}{n^{(m+1)/2}} \{ ||h^{(m)}||_{L_1(I)} + ||h||_{L_1(I)} + \frac{M}{n^{(2\beta-1)/2}} \},$$

where K is a constant independent of n and h.

Proof. With $F(t,u) = \sum_{j=0}^{m} \{ \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta^j h(u) \}$ as before, by Fubini's theorem we have

$$< P_{n,m}(h,t),g(t) > = \int_0^1 P_{n,m}(h,t)g(t)\,dt$$

$$= \int_0^1\int_0^1 K(n,t,u)\,F(t,u)g(t)\,du\,dt$$

$$(3.4.2) \qquad = \int_0^1\int_0^1 K(n,t,u)F(t,u)g(t)\,dt\,du.$$

Expanding $\Delta^j h(u)$ we have

$$F(t,u) = \sum_{j=0}^{m} \{ \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}}))(\sum_{r=0}^{j} \binom{j}{r} (-1)^{j-r} h(u+\frac{r}{n^{1/2}})) \}$$

$$= \sum_{j=0}^{m} \sum_{r=0}^{j} \{ \frac{n^{j/2}}{j!} \binom{j}{r} (-1)^{j-r} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) h(u+\frac{r}{n^{1/2}}) \}$$

$$= \sum_{r=0}^{m} h(u+\frac{r}{n^{1/2}}) \{ \sum_{j=r}^{m} \frac{n^{j/2}}{j!} \binom{j}{r} (-1)^{j-r} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \}$$

$$(3.4.3) = \sum_{r=0}^{m} h(u+\frac{r}{n^{1/2}})\, a_r(t,u), \text{ say.}$$

It follows from (3.4.2) and (3.4.3) that

$$\langle P_{n,m}(h,t), g(t) \rangle$$

$$= \sum_{r=0}^{m} \int_0^1 \int_0^1 K(n,t,u) h(u+\frac{r}{n^{1/2}}) a_r(t,u) g(t)\, dt\, du \quad .$$

For each r $(= 0,1,\ldots,m)$ we expand $g(t)$ in Taylor series about the point $u+\frac{r}{n^{1/2}}$.

$$g(t) = \sum_{i=0}^{m} \frac{1}{i!} (t-u-\frac{r}{n^{1/2}})^i g^{(i)}(u+\frac{r}{n^{1/2}})$$

$$+ \frac{1}{(m+1)!} (t-u-\frac{r}{n^{1/2}})^{m+1} g^{(m+1)}(\xi_r),$$

where ξ_r lies between $u+\frac{r}{n^{1/2}}$ and t.

Defining $h_i(u) = h(u)\, g^{(i)}(u)$, $0 \le i \le m$, we have

$$< P_{n,m}(h,t),g(t) >$$

$$= \sum_{i=0}^{m} \frac{1}{i!} \{ \sum_{r=0}^{m} \{ \int_0^1 \int_0^1 K(n,t,u) h_i(u+\frac{r}{n^{1/2}})(t-u-\frac{r}{n^{1/2}})^i a_r(t,u) dt du \} \}$$

$$+ \frac{1}{(m+1)!} \{ \sum_{r=0}^{m} \{ \int_0^1 \int_0^1 K(n,t,u) h(u+\frac{r}{n^{1/2}}) \, a_r(t,u) \times$$

$$\times \{ (t-u-\frac{r}{n^{1/2}})^{m+1} g^{(m+1)}(\xi_r) \} dt \, du \} \}.$$

This can be rewritten by (3.4.3) as

$$< P_{n,m}(h,t),g(t) >$$

$$= \sum_{i=0}^{m} \frac{1}{i!} \{ \int_0^1 \int_0^1 K(n,t,u) \times$$

$$\times \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\prod_{r=0}^{j-1} (t-u-\frac{r}{n^{1/2}})) \, \Delta^j((t-u)^i h_i(u)) \} dt \, du \}$$

$$+ \frac{1}{(m+1)!} \{ \sum_{r=0}^{m} \{ \int_0^1 \int_0^1 K(n,t,u) h(u+\frac{r}{n^{1/2}}) \, a_r(t,u) \times$$

$$\times \{ (t-u-\frac{r}{n^{1/2}})^{m+1} g^{(m+1)}(\xi_r) \} dt \, du \} \}$$

$$(3.4.4) \quad = \sum_{i=0}^{m+1} \frac{1}{i!} J_i, \text{ say.}$$

Firstly, a bound for J_{m+1} is obtained as follows. After expanding $a_r(t,u)$, a typical component of J_{m+1} can be written as

$$c \, n^{(j+k-m-s-1)/2} \int_0^1 \int_0^1 K(n,t,u)(t-u)^{j+k-s} h(u+\frac{r}{n^{1/2}}) g^{(m+1)}(\xi_r) dt du$$

$$= T, \text{ say,}$$

where $0 \leq r \leq m$, $r \leq j \leq m$, $0 \leq k \leq m+1$, $0 \leq s \leq j-1$ and c is a scalar. Since supp $h \subset (0,1)$ it follows from Proposition 2.1.1 and the boundedness of $g^{(m+1)}$ that

$$|T| \leq \frac{M_1}{n^{(m+1)/2}} \, ||h||_{L_1(I)}.$$

Thus,

$$(3.4.5) \qquad |J_{m+1}| \leq \frac{M_2}{n^{(m+1)/2}} \, ||h||_{L_1(I)}.$$

Next, for J_i where $1 \leq i \leq m$, using Fubini's theorem we obtain

$$J_i = \int_0^1\int_0^1 K(n,t,u) \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta^j((t-u)^i h_i(u))\} du dt.$$

Since $h_i(u)$ can be expanded as

$$h_i(u) = \sum_{r=0}^{m-i} \frac{(u-t)^r}{r!} h_i^{(r)}(t) + \frac{1}{(m-i)!} \int_t^u (u-w)^{m-i} h_i^{(m+1-i)}(w)\, dw,$$

it follows from (i) of Lemma 3.1.1 that

$$J_i = \frac{1}{(m-i)!} \int_0^1\int_0^1 K(n,t,u) \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \times$$

$$\times (\Delta^j((t-u)^i \int_t^u (u-w)^{m-i} h_i^{(m+1-i)}(w) dw))\} du\, dt.$$

Proceeding as for the estimate of J_1 in Theorem 3.2.3 we obtain

$$|J_i| \leq \frac{M_3}{n^{(m+1)/2}} \, ||h_i^{(m+1-i)}||_{L_1(I)}.$$

Further, applying Lemma 1.2.2

$$(3.4.6) \quad |J_i| \leq \frac{M_4}{n^{(m+1)/2}} (||h^{(m)}||_{L_1(I)} + ||h||_{L_1(I)}).$$

Lastly, we evaluate J_o. To do so we require the following auxiliary results.

<u>Lemma 3.4.3.</u> Let $h \in C_o$ have a compact support $\subset (a,b)$, $(a,b \in \mathbb{R})$. Then, for $k, r \in \mathbb{N}^o$ and for sufficiently small $\delta > 0$

$$(3.4.7)\quad \int_a^b y^k \Delta_\delta^r\, h(y)dy = \begin{cases} (-\delta)^r\, r!\, (\int_a^b h(y)dy), & k = r, \\ 0, & k < r. \end{cases}$$

<u>Proof</u>. By the definition of $\Delta_\delta^r\, h(y)$ we have

$$\int_a^b y^k \Delta_\delta^r\, h(y)dy = \sum_{i=0}^{r} \binom{r}{i}(-1)^{r-i} \{\int_a^b y^k\, h(y+i\delta)\, dy\} .$$

Since supp $h \subset (a,b)$, for all sufficiently small δ we have

$$\int_a^b y^k\, \Delta_\delta^r\, h(y)dy = \sum_{i=0}^{r} \binom{r}{i}(-1)^{r-i} \{\int_a^b (y-i\delta)^k\, h(y)\, dy\}$$

$$= \sum_{i=0}^{r} \binom{r}{i}(-1)^{r-i} \{\sum_{j=0}^{k} \binom{k}{j}(-i\delta)^{k-j}(\int_a^b y^j\, h(y)dy)\}$$

$$= \sum_{j=0}^{k} \binom{k}{j} \{\sum_{i=0}^{r} \binom{r}{i}(-1)^{r-i}\, (-i\delta)^{k-j}\}(\int_a^b y^j h(y)dy).$$

Now using the binomial identity

$$\sum_{i=0}^{r} \binom{r}{i}(-1)^i\, i^m = \begin{cases} (-1)^r\, r! \;, & m = r, \\ 0 \;, & m < r, \end{cases}$$

we obtain

$$\int_a^b y^k\, \Delta_\delta^r\, h(y)dy = \begin{cases} (-\delta)^r\, r!\, (\int_a^b h(y)\, dy), & k = r \\ 0 \;, & k < r. \end{cases}$$

Lemma 3.4.4. Let $j \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $|x|$ be sufficiently small. Then in the expansion of $(1+x)^{j\alpha}/\prod_{i=1}^{j}(1+ix)$ in the powers of x :

$$(3.4.8)\qquad \frac{(1+x)^{j\alpha}}{\prod_{i=1}^{j}(1+ix)} = 1 + P_1(j)\,x + P_2(j)\,x^2 + \dots,$$

the $P_k(j)$ are polynomials in j of degree $2k$.

Proof. Taking logarithm of both sides in (3.4.8) we obtain

$$j\alpha\ \ln(1+x) - \sum_{i=1}^{j} \ln(1+ix) = \ln\{1+P_1(j)x+P_2(j)x^2 + \dots\}.$$

Using the series expansion this reduces to

$$-j\alpha\Big(\sum_{k=1}^{\infty} \frac{(-x)^k}{k}\Big) + \sum_{i=1}^{j}\Big(\sum_{k=1}^{\infty} \frac{(-ix)^k}{k}\Big) = \Big(\sum_{k=1}^{\infty} P_k(j)x^k\Big) - \frac{1}{2}\Big(\sum_{k=1}^{\infty} P_k(j)x^k\Big)^2 + \frac{1}{3}\Big(\sum_{k=1}^{\infty} P_k(j)x^k\Big)^3 + \dots .$$

Collecting coefficients of powers of x on both sides we obtain

$$\Big(\sum_{k=1}^{\infty} \frac{(-x)^k}{k}\Big)\Big(-j\alpha + \sum_{i=1}^{j} i^k\Big) = P_1(j)x + x^2\Big(P_2(j) - \frac{1}{2}P_1^2(j)\Big)$$

$$+ x^3\{P_3(j) - P_1(j)P_2(j) + \frac{1}{3}P_1^3(j)\} + \dots$$

$$+ x^k\{P_k(j) - \frac{1}{2}\sum_{p+q=k} P_p(j)P_q(j) +$$

$$+ \frac{1}{3}\sum_{p+q+r=k} P_p(j)P_q(j)P_r(j) + \dots +$$

$$+ \frac{(-1)^{k-1}}{k} P_1^k(j)\} + \dots .$$

Using the well known fact that for $k \in \mathbb{N}$, $\sum_{i=1}^{j} i^k$ is a polynomial in j of degree $k+1$ and comparing the coefficients of like powers of x the proof follows.

Putting $\alpha = 0$ in Lemma 3.4.4 we obtain the following :

Corollary 3.4.5. For $j \in \mathbb{N}$ and $|x| < j^{-1}$

$$\prod_{i=1}^{j} (1+ix)^{-1} = 1 + Q_1(j)x + Q_2(j)x^2 + \dots,$$

where $Q_k(j)$ is a polynomial in j of degree $2k$.

Resuming the proof of Lemma 3.4.2, writing

$$\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}}) = (t-u)^j + \sum_{r=1}^{j-1} \frac{d_{j,r}}{n^{r/2}} (t-u)^{j-r},$$

where $d_{j,r}$'s are constants,

we have from (3.4.4)

$$J_o = \int_o^1\int_o^1 K(n,t,u) \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta^j h_o(u)\} dt\, du$$

$$= \sum_{j=0}^{m} \frac{n^{j/2}}{j!} \{\int_o^1\int_o^1 K(n,t,u)(t-u)^j \Delta^j h_o(u)\, dt\, du\}$$

$$+ \sum_{j=2}^{m} \sum_{r=1}^{j-1} \frac{d_{j,r}}{j!} n^{(j-r)/2} \{\int_o^1\int_o^1 K(n,t,u)(t-u)^{j-r} \Delta^j h_o(u) dt\, du\}.$$

Now

$$\int_o^1 p_{n\nu}(t)(t-u)^k\, dt = \sum_{s=0}^{k} \binom{k}{s}(-1)^s u^s (\int_o^1 p_{n\nu}(t) t^{k-s}\, dt)$$

This gives that

$$\int_0^1 p_{n\nu}(t)(t-u)^k\,dt = \sum_{s=0}^{k} \binom{k}{s}(-1)^s u^s \left\{ \frac{\prod_{i=1}^{k-s} (\nu+i)}{\prod_{i=1}^{k-s+1} (n+i)} \right\},$$

where $\prod_{i=1}^{j} (\nu+i)$ is to be interpreted as 1 when $j = 0$. We have, after writing $K(n,t,u) = (n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t)\, \chi_{n\nu}(u)\}$,

$$J_o = (n+1)\{\sum_{j=0}^{m} \sum_{s=0}^{j} \binom{j}{s} \frac{(-1)^s}{j!} \frac{n^{j/2}}{\prod_{i=1}^{j+1-s} (n+i)} \times$$

$$\times \{\sum_{\nu=0}^{n} \{(\prod_{i=1}^{j-s} (\nu+i)) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^s\, \Delta^j h_o(u)du\}\}\}$$

$$+ (n+1)\{\sum_{j=2}^{m} \sum_{r=1}^{j-1} \sum_{s=0}^{j-r} \binom{j-r}{s} (-1)^s \frac{d_{j,r}}{j!} \frac{n^{(j-r)/2}}{\prod_{i=1}^{j+1-r-s} (n+i)} \times$$

$$\text{(3.4.9)} \qquad \times \{\sum_{\nu=0}^{n} (\prod_{i=1}^{j-r-s} (\nu+i)) \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^s\, \Delta^j h_o(u)du\}\}.$$

We see from above that a typical component of J_o is of the type

$$c(n+1) \frac{n^{r_1/2}}{\prod_{i=1}^{r_2} (n+i)} \{\sum_{\nu=0}^{n} \nu^{r_3} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^{r_4}\, \Delta^{r_5} h_o(u)du\},$$

where c is a constant and r_i $(i = 1,2,\ldots,5) \in \mathbb{N}^0$ are such that $0 \le r_1 \le m$, $1 \le r_2 \le m+1$, $r_3+r_4 \le r_5$, $0 \le r_5 \le m$.

Next, using Euler-Maclaurin summation formula (Lemma 1.7.10) we change the summation

$$\sum_{\nu=0}^{n} \nu^{r_3} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^{r_4}\, \Delta^{r_5} h_o(u)\, du$$

into an approximate integral as follows.

Writing $H(x) = x^{r_3} \int_x^{x+1/(n+1)} u^{r_4} \Delta^{r_5} h_o(u)\,du$,

it follows from the given smoothness hypothesis on the function h that $H(x)$ is m+1 times differentiable with $H^{(m)} \in A.C.(I)$ and $H^{(m+1)} \in L_p(I)$. Also supp $H \subset (0,1)$ for all n sufficiently large. Then, by Lemma 1.7.10, we have

$$\sum_{\nu=0}^{n} \nu^{r_3} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^{r_4} \Delta^{r_5} h_o(u)du = (n+1)^{r_3}\left(\sum_{\nu=0}^{n} H\left(\frac{\nu}{n+1}\right)\right)$$

$$= (n+1)^{r_3+1}\left\{\int_0^1 H(x)dx - R_{r_3,r_4,r_5}\right\}, \tag{3.4.10}$$

where R_{r_3,r_4,r_5} is given as follows :

$$R_{r_3,r_4,r_5} = \begin{cases} \dfrac{1}{(n+1)^{2k+1}}\left\{\sum_{r=0}^{n}\left\{\int_0^1 P_{2k}(t)H^{(2k)}\left(\frac{t+r}{n+1}\right)dt\right\}\right\}, & m = 2k-1, \\[2ex] -\dfrac{1}{(n+1)^{2k+2}}\left\{\sum_{r=0}^{n}\left\{\int_0^1 P_{2k+1}(t)H^{(2k+1)}\left(\frac{t+r}{n+1}\right)\right\}\right\}, & m = 2k. \end{cases} \tag{3.4.11}$$

Let $\chi_x(u)$ denote the characteristic function of $\left[x, x+\frac{1}{n+1}\right]$

$$\text{Then,} \int_0^1 H(x)\,dx = \int_0^1 x^{r_3} \int_x^{x+1/(n+1)} u^{r_4} \Delta^{r_5} h_o(u)\,du\,dx$$

$$= \int_0^1\int_0^1 \chi_x(u) x^{r_3} u^{r_4} \Delta^{r_5} h_o(u)\,du\,dx.$$

Interchanging integrals by Fubini's theorem we obtain

$$\int_0^1 H(x)dx = \int_0^1\int_0^1 \chi_x(u) x^{r_3} u^{r_4} \Delta^{r_5} h_o(u)dxdu$$

$$= \int_0^1 u^{r_4} \Delta^{r_5} h_o(u)\left(\int_0^1 \chi_x(u)\, x^{r_3}\, dx\right)du$$

$$= \int_0^1 u^{r_4} \Delta^{r_5} h_o(u) \left(\int_{u-1/(n+1)}^{u} x^{r_3}\, dx\right) du$$

$$= -\frac{1}{(r_3+1)} \left\{ \sum_{s=0}^{r_3} \binom{r_3+1}{s}\left(-\frac{1}{n+1}\right)^{r_3+1-s} \left(\int_0^1 u^{r_4+s} \Delta^{r_5} h_o(u)du\right)\right\} .$$

By Lemma 3.4.3 the expression on the right side can be written as

$$(3.4.12)\quad \int_0^1 H(x)dx = \begin{cases} (-1)^{r_5} r_5!\, \frac{n^{-r_5/2}}{(n+1)} \left(\int_0^1 h_o(u)du\right) , & r_3+r_4=r_5, \\ 0 & , r_3+r_4 < r_5. \end{cases}$$

Combining (3.4.10) and (3.4.12) we obtain

$$\sum_{\nu=0}^{n} \nu^{r_3} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^{r_4} \Delta^{r_5} h_o(u)\, du =$$

$$(3.4.13)\quad \begin{cases} (-1)^{r_5} r_5!\; n^{-r_5/2} (n+1)^{r_3} \left(\int_0^1 h_o(u)du\right) \\ \qquad\qquad -(n+1)^{r_3+1} R_{r_3,r_4,r_5} & ,\text{if } r_3+r_4 = r_5 \\ -(n+1)^{r_3+1} R_{r_3,r_4,r_5} , & ,\text{if } r_3+r_4 < r_5 \end{cases}$$

where R_{r_3,r_4,r_5} is given by (3.4.11).

Thus (3.4.13) gives a formula for changing the summation

$$\sum_{\nu=0}^{n} \{ \nu^{r_3} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^{r_4} \Delta^{r_5} h_o(u)\, du \}$$

into an approximate integral with an error term. We use it to obtain an approximate integral for J_o as follows :

Writing

$$\prod_{i=1}^{k} (\nu+i) = \nu^{k} + \sum_{i=1}^{k} b_{k,i}\, \nu^{k-i} ,$$

where $b_{k,i}$ are certain constants.

We have from (3.4.9)

$$J_o = (n+1) \{ \sum_{j=0}^{m} \sum_{s=0}^{j} \binom{j}{s} \frac{(-1)^s}{j!} \frac{n^{j/2}}{\prod_{i=1}^{j+1-s} (n+i)} \times$$

$$\times \{ \sum_{\nu=0}^{n} \{ \nu^{j-s} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^s \Delta^j h_o(u) du \}\}$$

$$+ (n+1) \{ \sum_{j=0}^{m} \sum_{s=0}^{j-1} \sum_{k=1}^{j-s} \binom{j}{s} \frac{(-1)^s}{j!} \frac{n^{j/2}}{\prod_{i=1}^{j+1-s} (n+i)} b_{j-s,k} \times$$

$$\times \{ \sum_{\nu=0}^{n} \{ \nu^{j-s-k} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^s \Delta^j h_o(u)\, du \}\} +$$

$$+ \ (n+1) \{ \sum_{j=2}^{m} \sum_{r=1}^{j-1} \sum_{s=0}^{j-r} \binom{j-r}{s} \frac{(-1)^s}{j!} \frac{n^{(j-r)/2}}{\prod_{i=1}^{j+1-r-s} (n+1)} d_{j,r} \times$$

$$\times \{ \sum_{\nu=0}^{n} \{ \nu^{j-r-s} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^s \Delta^j h_o(u) du \}\}\}$$

$$+ \ (n+1) \{ \sum_{j=2}^{m} \sum_{r=1}^{j-1} \sum_{s=0}^{j-r-1} \sum_{k=1}^{j-r-s} \binom{j-r}{s} \frac{(-1)^s}{j!} \frac{n^{(j-r)/2}}{\prod_{i=1}^{j+1-r-s} (n+i)} d_{j,r} \times$$

$$\times \ b_{j-r-s,k} \{ \sum_{\nu=0}^{n} \{ \nu^{j-r-s-k} \int_{\nu/(n+1)}^{(\nu+1)/(n+1)} u^s \Delta^j h_o(u) du \}\}\}$$

$(3.4.14) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$, say.

We see from (3.4.13) that Σ_2, Σ_3 and Σ_4 consist of only remainder terms involving R_{r_3,r_4,r_5}.

Applying (3.4.13) to Σ_1 we obtain

$$\Sigma_1 = (n+1) \{ \sum_{j=0}^{m} \{ \sum_{s=0}^{j} \binom{j}{s} \frac{(-1)^s}{j!} \frac{n^{j/2}}{\prod_{i=1}^{j+1-s} (n+i)} \} \times$$

$$\times \{ (-1)^j (n+1)^{j-s} (n)^{-j/2} j! (\int_0^1 h_o(u) du) - (n+1)^{j+1-s} R_{j-s,s,j} \}\}$$

$$= \{ \sum_{j=0}^{m} \sum_{s=0}^{j} \{ \binom{j}{s} (-1)^{j+s} \{ \frac{(n+1)^{j+1-s}}{\prod_{i=1}^{j+1-s} (n+i)} \}\}\} (\int_0^1 h_o(u) du)$$

$$- \sum_{j=0}^{m} \sum_{s=0}^{j} \{ \binom{j}{s} \frac{(-1)^s}{j!} n^{j/2} \{ \frac{(n+1)^{j+2-s}}{\prod_{i=1}^{j+1-s} (n+i)} \} \} R_{j-s,s,j}$$

(3.4.15) $= \Sigma_{11} - \Sigma_{12}$, say.

Rearranging the terms in Σ_{11} we obtain

$$\Sigma_{11} = (\int_0^1 h_0(u)du) \{ \sum_{j=0}^{m} \sum_{s=0}^{j} \binom{j}{s} (-1)^{j+s} \{ \frac{(n+1)^{j+1-s}}{\prod_{i=1}^{j+1-s} (n+i)} \} \}$$

$$= (\int_0^1 h_0(u)du) \{ \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{j}{j-k} (-1)^k \{ \frac{(n+1)^{k+1}}{\prod_{i=1}^{k+1} (n+i)} \} \}$$

$$= (\int_0^1 h_0(u)du) \{ \sum_{k=0}^{m} (-1)^k \{ \frac{(n+1)^{k+1}}{\prod_{i=1}^{k+1} (n+i)} \} (\sum_{j=k}^{m} \binom{j}{j-k}) \}$$

$$= (\int_0^1 h_0(u)du) \{ \sum_{k=0}^{m} (-1)^k \binom{m+1}{k+1} \{ \frac{(n+1)^{k+1}}{\prod_{i=1}^{k+1} (n+i)} \} \}$$

(where we have used the binomial identity

$$\sum_{j=k}^{m} \binom{j}{k} = \binom{m+1}{k+1} \;)$$

$$= - \{ \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} \{ \frac{(n+1)^k}{\prod_{i=1}^{k} (n+i)} \} \} (\int_0^1 h_0(u)du) + \int_0^1 h_0(u)du,$$

where $\prod_{i=1}^{k} (n+i)$ for $k = 0$ is to be interpreted as 1.

Thus using Lemma 3.4.4 and the binomial identity

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^i \, i^m = \begin{cases} (-1)^r \, r!, & m = r \\ 0, & m < r \end{cases}$$

we get,

$$(3.4.16) \quad \Sigma_{11} = \int_0^1 h_o(u)du + ||h||_{L_1(I)} \, O\left(\frac{1}{n^{[\frac{m}{2}]+1}}\right), \quad (n \to \infty).$$

Now we obtain a bound for the remainder terms of Σ_{12}, Σ_2, Σ_3 and Σ_4.

We see from (3.4.11), (3.4.14) and (3.4.15) that typical remainder terms of Σ_{12}, Σ_2, Σ_3 and Σ_4 can be written, respectively, as

$$c_1 \, n^{j/2} \frac{(n+1)^{j+2-s}}{\prod_{i=1}^{j+1-s} (n+i)} R_{j-s,s,j}, \quad (0 \leq s \leq j),$$

$$c_2 \, n^{j/2} \frac{(n+1)^{j+2-k-s}}{\prod_{i=1}^{j+1-s} (n+i)} R_{j-s-k,s,j}, \quad (0 \leq s \leq j-1, 1 \leq k \leq j-s),$$

(3.4.17)

$$c_3 n^{(j-r)/2} \frac{(n+1)^{j+2-r-s}}{\prod_{i=1}^{j+1-r-s} (n+i)} R_{j-r-s,s,j},$$

$$(1 \leq r \leq j-1, \; 0 \leq s \leq j-r),$$

$$c_4 n^{(j-r)/2} \frac{(n+1)^{j+2-r-s-k}}{\prod_{i=1}^{j+1-r-s} (n+i)} R_{j-r-s-k,s,j},$$

$$(1 \leq r \leq j-1, 0 \leq s \leq j-r-1, 1 \leq k \leq j-r-s),$$

where c_i's are constants.

We analyse the error terms occuring in Σ_{12}, Σ_2, Σ_3 and Σ_4. We investigate separately the two cases depending on whether m is an even or an odd integer. For this we take up a typical term given by (3.4.17). Using the condition

$$\omega(h^{(m)},\tau,p,I) \leq M\tau^{\beta}, \quad (\tau \to 0),$$

we show that any such typical term is bounded by

$$\frac{K}{n^{(m+1)/2}} \left(\|h^{(m)}\|_{L_1(I)} + \|h\|_{L_1(I)} + \frac{M}{n^{(2\beta-1)/2}} \right).$$

Case I. Let m be an odd integer, say 2k-1. Defining $\psi(u) = u^{r_4} \Delta^{r_5} h_o(u)$, we see that the function H(x) defined as

$$H(x) = x^{r_3} \int_x^{x+1/(n+1)} \psi(u)\, du$$

is 2k times differentiable. Hence for almost all values of x

$$H^{(2k)}(x) = x^{r_3} \Delta_{1/(n+1)} \psi^{(2k-1)}(x) + \binom{2k}{1}(r_3 x^{r_3-1}) \Delta_{1/(n+1)} \psi^{(2k-2)}(x)$$

$$+ \dots + \binom{2k}{r_3}(r_3!)\, \Delta_{1/(n+1)} \psi^{(2k-r_3-1)}(x). \tag{3.4.18}$$

By definition we have

$$\Delta_{1/(n+1)} \psi(x) = \Delta_{1/(n+1)} (x^{r_4} \Delta^{r_5} h_o(x))$$

$$= \left(x + \frac{1}{n+1}\right)^{r_4} (\Delta_{1/(n+1)} \Delta^{r_5} h_o(x))$$

$$+ (\Delta_{1/(n+1)} x^{r_4})(\Delta^{r_5} h_o(x)).$$

Also using the fact that for any $\delta_1, \delta_2 > 0$,

$$\Delta_{\delta_1} \Delta_{\delta_2} f(x) = \Delta_{\delta_2} \Delta_{\delta_1} f(x),$$

we obtain

$$(3.4.19) \quad \Delta_{1/(n+1)} \psi(x) = (x + \tfrac{1}{n+1})^{r_4} (\Delta^{r_5} \Delta_{1/(n+1)} h_o(x)) + (\Delta_{1/(n+1)} x^{r_4}) (\Delta^{r_5} h_o(x)).$$

Differentiating the above identity r times we obtain

$$\Delta_{1/(n+1)} \psi^{(r)}(x) = (\Delta_{1/(n+1)} \psi(x))^{(r)}$$

$$= ((x+\tfrac{1}{n+1})^{r_4}(\Delta^{r_5} \Delta_{1/(n+1)} h_o(x))^{(r)} + ((\Delta_{1/(n+1)} x^{r_4})(\Delta^{r_5} h_o(x))^{(r)}$$

$$= \{(\Delta^{r_5} \Delta_{1/(n+1)} h_o^{(r)}(x)) (x + \tfrac{1}{n+1})^{r_4} + \binom{r}{1}(\Delta^{r_5} \Delta_{1/(n+1)} h_o^{(r-1)}(x)) \times$$

$$\times \{r_4(x + \tfrac{1}{n+1})^{r_4-1}\} + \cdot\cdot + \binom{r}{r_4}(\Delta^{r_5} \Delta_{1/(n+1)} h_o^{(r-r_4)}(x))(r_4!)\}$$

$$+ \{(\Delta^{r_5} h_o^{(r)}(x))(\Delta_{1/(n+1)} x^{r_4}) + \binom{r}{1}(\Delta^{r_5} h_o^{(r-1)}(x)) r_4 (\Delta_{1/(n+1)} x^{r_4-1})$$

$$(3.4.20) \qquad + \cdot\cdot + \binom{r}{r_4-1}(\Delta^{r_5} h_o^{(r-r_4+1)}(x)) \frac{r_2!}{n+1} \}.$$

Thus, we see from (3.4.18) and (3.4.20) that $H^{(2k)}(x)$ consists of terms which involve $\Delta_{1/(n+1)} h_o^{(2k-1)}(x)$, $\Delta h_o^{(2k-1)}(x)$ and Δ-differences of the derivatives of $h_o(x)$ lower than $h_o^{(2k-1)}(x)$.

Now, from (3.4.11) it follows that

$$R_{r_3,r_4,r_5} \le \frac{1}{(n+1)^{2k+1}} ||P_{2k}||_{C(I)} \{ \sum_{r=0}^{n} \int_0^1 |H^{(2k)}(\tfrac{t+r}{n+1})| \, dt \}$$

$$(3.4.21) \qquad \le \frac{1}{(n+1)^{2k}} ||P_{2k}||_{C(I)} \, ||H^{(2k)}||_{L_1(I)}.$$

The terms which involve highest derivative of h_o i.e., $h_o^{(2k-1)}(x)$ in $H^{(2k)}(x)$ are :

$$x^{r_3}\{(\Delta^{r_5} \Delta_{1/(n+1)} h_o^{(2k-1)}(x))(x+\tfrac{1}{n+1})^{r_4} +$$

$$+ (\Delta^{r_5} h_o^{(2k-1)}(x)) \, (\Delta_{1/(n+1)} \, x^{r_4})\}$$

$$(3.4.22) \qquad = T(x), \text{ say.}$$

We will obtain a bound for $||T||_{L_1(I)}$ by making use of the hypothesis on $h^{(2k-1)}$:

$$(3.4.23) \qquad \omega(h^{(2k-1)}, \tau, p, I) \le M\tau^{\beta} \;, \quad (\tau \to 0).$$

For this, we shall first prove that

$$(3.4.24) \qquad \omega(h_o^{(2k-1)}, \tau, p, I) = O(\tau^{\beta}), \quad (\tau \to 0).$$

Since $h_o(x) = h(x)\, g(x)$, where $h^{(2k-2)} \in A.C.(I)$, $h^{(2k-1)} \in L_p(I)$ and $g \in C^{2k}(I)$, one obtains for almost all values of x

$$h_o^{(2k-1)}(x) = h^{(2k-1)}(x) g(x) + \binom{2k-1}{1} h^{(2k-2)}(x) g^{(1)}(x) + \ldots$$

$$+ h(x)\, g^{(2k-1)}(x).$$

From this it follows that for almost all values of x

$$\Delta_{1/(n+1)} h_o^{(2k-1)}(x) = \Delta_{1/(n+1)}(h^{(2k-1)}(x)g(x))$$

$$+ \binom{2k-1}{1} \Delta_{1/(n+1)} (h^{(2k-2)}(x)\, g^{(1)}(x)) + \ldots$$

$$(3.4.25) \qquad + \Delta_{1/(n+1)} (h(x)\, g^{(2k-1)}(x)).$$

In view of the fact that $\Delta_{1/(n+1)} (h^{(2k-1)}(x)g(x))$

$$= (\Delta_{1/(n+1)} h^{(2k-1)}(x))g(x+\tfrac{1}{n+1}) + h^{(2k-1)}(x)(\Delta_{1/(n+1)} g(x))$$

$$= (\Delta_{1/(n+1)} h^{(2k-1)}(x))g(x+\tfrac{1}{n+1}) + \tfrac{1}{n+1}\, h^{(2k-1)}(x)\, g'(\xi),$$

(where ξ lies between x and $x + \frac{1}{n+1}$)

and (3.4.23), we obtain

$$(3.4.26) \quad ||\Delta_{1/(n+1)}(h^{(2k-1)}(x)g(x))||_{L_1(I)}$$

$$\leq M_5(\frac{M}{n^\beta} + \frac{1}{n}||h^{(2k-1)}||_{L_1(I)}).$$

This gives an L_1-estimate of the first term on the right side of the expression (3.4.25).

Other terms in (3.4.25) are of the type

$$c\, \Delta_{1/(n+1)} (h^{(2k-1-r)}(x)\, g^{(r)}(x)),$$

where $1 \leq r \leq 2k-1$ and c is a constant. This is nothing but

$$c\ \{(\Delta_{1/(n+1)}\ h^{(2k-1-r)}(x))g^{(r)}(x+\tfrac{1}{n+1})$$

$$+\ h^{(2k-1-r)}(x)\ (\Delta_{1/(n+1)}\ g^{(r)}(x))\ \}.$$

Hence, $||c\ \Delta_{1/(n+1)}\ (h^{(2k-1-r)}(x)\ g^{(r)}(x))||_{L_1(I)}$

$$(3.4.27)\qquad \leq M_6\{\int_0^1 |\Delta_{1/(n+1)}h^{(2k-1-r)}(x)|dx+\tfrac{1}{n}||h^{(2k-1-r)}||_{L_1(I)}\}.$$

To obtain a bound for the first term on the right side of the above inequality (3.4.27) we write

$$\int_0^1 |\Delta_{1/(n+1)}\ h^{(2k-1-r)}(x)|dx = \int_0^1|\int_x^{x+1/(n+1)} h^{(2k-r)}(y)dy|dx$$

$$\leq \int_0^1\int_0^{x+1/(n+1)}|h^{(2k-r)}(y)|dy\ dx.$$

Let $\chi_x(y)$ be the characteristic function of $[x,x+1/(n+1)]$.

Then

$$\int_0^1 |\Delta_{1/(n+1)}h^{(2k-1-r)}(x)|dx \leq \int_0^1\int_0^1 \chi_x(y)|h^{(2k-r)}(y)|dy\ dx$$

$$= \int_0^1\int_0^1 \chi_x(y)|h^{(2k-r)}(y)|dx\ dy \text{ (By Fubini's theorem)}$$

$$= \int_0^1 |h^{(2k-r)}(y)|(\int_0^1 \chi_x(y)dx)dy$$

$$(3.4.28)\qquad = \tfrac{1}{n+1}\ ||h^{(2k-r)}||_{L_1(I)}.$$

Thus, from (3.4.27) and (3.4.28) we obtain that

$$(3.4.29)\qquad \|c\,\Delta_{1/(n+1)}(h^{(2k-1-r)}(x)g^{(r)}(x))\|_{L_1(I)}$$

$$\leq \frac{M_6}{n}\{\|h^{(2k-r)}\|_{L_1(I)} + \|h^{(2k-1-r)}\|_{L_1(I)}\}.$$

Finally, we obtain from (3.4.25), (3.4.26) and (3.4.29)

$$(3.4.30)\qquad \|\Delta_{1/(n+1)}\, h_o^{(2k-1)}(x)\|_{L_1(I)}$$

$$\leq M_7\{\frac{M}{n^\beta} + \frac{1}{n}(\sum_{j=0}^{2k-1}\|h^{(j)}\|_{L_1(I)})\},$$

which proves (3.4.24).

This, in conjunction with (3.4.22), proves that

$$(3.4.31)\qquad \|T(x)\|_{L_1(I)} \leq M_8\{\frac{M}{n^\beta} + \frac{1}{n}(\sum_{j=0}^{2k-1}\|h^{(j)}\|_{L_1(I)})\}.$$

Regarding the other terms of $H^{(2k)}(x)$, which involve lower derivatives of h_o, we proceed as in the above. For instance, in $H^{(2k)}(x)$, terms involving $h_o^{(2k-2)}(x)$ are :

$$x^{r_3}\binom{2k-1}{1}\{(\Delta^{r_5}(\Delta_{1/(n+1)}h_o^{(2k-2)}(x))r_4(x+\frac{1}{n+1})^{r_4-1}$$

$$+ (\Delta^{r_5}\, h_o^{(2k-2)}(x))\;\; r_4(\Delta_{1/(n+1)}\, x^{r_4-1})\}$$

$$+ 2kr_3\, x^{r_3-1}\{(\Delta^{r_5}\,\Delta_{1/(n+1)}\, h_o^{(2k-2)}(x))(x+\frac{1}{n+1})^{r_4}$$

$$+ (\Delta^{r_5}\, h_o^{(2k-2)}(x))\,(\Delta_{1/(n+1)}\, x^{r_4})\}$$

$$= T_1(x),\ \text{say}.$$

It follows after applying the estimate (3.4.28) and the fact $\Delta_{1/(n+1)}\, x^m = O(n^{-1})$, $(n \to \infty)$, that

$$(3.4.32) \qquad \|T_1(x)\|_{L_1(I)} \leq \frac{M_9}{n}\Big(\sum_{j=0}^{2k-1} \|h^{(j)}\|_{L_1(I)}\Big).$$

The estimates (3.4.21), (3.4.31) and (3.4.32) give

$$(3.4.33) \quad |R_{r_3,r_4,r_5}| \leq \frac{M_{10}}{n^{2k}}\Big\{\frac{M}{n^\beta} + \frac{1}{n}\Big(\sum_{j=0}^{2k-1} \|h^{(j)}\|_{L_1(I)}\Big)\Big\}.$$

The case when m is an even integer, is treated analogously and we obtain similar estimates.

Having obtained a bound for R_{r_3,r_4,r_5} we see from (3.4.17) and (3.4.33) that when $m = 2k-1$ any general term occuring in (3.4.17), say R, has the following bound

$$|R| \leq M_{11}\, n^{(2k-1)/2}\, n\, |R_{r_3,r_4,r_5}|$$

$$(3.4.34) \qquad \leq M_{12}\, \frac{n^{1/2}}{n^k}\Big\{\frac{M}{n^\beta} + \frac{1}{n}\Big(\sum_{j=0}^{2k-1} \|h^{(j)}\|_{L_1(I)}\Big)\Big\}.$$

In view of Lemma 1.2.2 this is further bounded as :

$$(3.4.35) \quad |R| \leq \frac{M_{13}}{n^k}\Big\{\frac{M}{n^{(2\beta-1)/2}} + \frac{1}{n^{1/2}}\Big(\|h^{(2k-1)}\|_{L_1(I)} + \|h\|_{L_1(I)}\Big)\Big\}.$$

Finally, collecting together (3.4.4), (3.4.5), (3.4.6), (3.4.14), (3.4.15), (3.4.16), (3.4.17) and (3.4.35) we obtain for odd integral values of m

$$|< P_{n,m}(h,t)-h(t),g(t) >| \leq \frac{K}{n^{(m+1)/2}} \{ \frac{M}{n^{(2\beta-1)/2}} + ||h^{(m)}||_{L_1(I)}$$

$$+ ||h||_{L_1(I)} \} .$$

This proves that lemma.

Proof of the theorem. We proceed exactly as in the proof of Theorem 2.4.1. It follows from Theorem 3.3.1 and Theorem 1.3.2 that for every subinterval $[c,d] \subset (a_1,b_1)$, f coincides a.e. on $[c,d]$ with a function F possessing an absolutely continuous derivative $F^{(m-1)}$, and an mth derivative $F^{(m)}$ which belongs to $L_p[c,d]$. Moreover, for $0 < \beta < 1$

$$\omega(F^{(m)},\tau,p,I) \leq M \tau^{\beta} \quad , \quad (\tau \to 0).$$

We choose pairs of points x_1,x_2 and y_1,y_2 such that $a_1 < x_1 < x_2 < a_2 < b_2 < y_2 < y_1 < b_1$. Also let $q \in C_o^{m+1}$ with supp $q \subset (a_1,b_1)$ and $q(t) = 1$ if $t \in [x_1,y_1]$. Then by Theorem 3.1.3 and statement (i) of Theorem 3.4.1 we have for $G(u) = F(u)q(u)$

$$||P_{n,m}(G,t)-G(t)||_{L_p[x_2,y_2]} = O(n^{-(m+1)/2}), \ (n \to \infty).$$

As in (2.4.9) there is a function $H(t) \in L_p[x_2,y_2]$ $(p > 1)$ such that for some subsequence $\{n_j\}$ and for every $g \in C_o^{m+1}$ with supp $g \subset (0,1)$ we have

(3.4.36) $$\lim_{n_j \to \infty} n_j^{(m+1)/2} < P_{n_j,m}(G,t)-G(t),g(t) > = < H(t),g(t) >.$$

When $p = 1$ as in (2.4.12), there exists a function $\phi_o(t) \in B.V.\,[x_2,y_2]$ such that for every $g \in C_o^{m+1}$ with $\operatorname{supp} g \subset (x_2,y_2)$ and for some subsequence $\{n_j\}$ we have

$$(3.4.37)\qquad \lim_{n_j \to \infty} n_j^{(m+1)/2} \langle P_{n_j,m}(G,t)-G(t), g(t) \rangle$$

$$= -\langle \phi_o(t), g'(t) \rangle.$$

Let $G_{n,m+1}$ be the Steklov mean of $(m+1)$th order corresponding to G. Writing

$$P_{m+1}(D)(\cdot,t) = \frac{(-1)^m}{(m+1)!}\, p_{m+1}(t)\, \frac{d^{m+1}}{dt^{m+1}}(\cdot,t),$$

we obtain by Theorem 3.2.6,

$$(3.4.38)\qquad P_{n_j,m}(G_{n,m+1},t)-G_{n,m+1}(t)$$

$$= \frac{1}{n_j^{(m+1)/2}} P_{m+1}(D)G_{n,m+1}(t) + o\left(\frac{1}{n_j^{(m+1)/2}}\right),$$

where the o-term may depend on n but for each fixed n it holds uniformly in $t \in I_1$.

We obtain from (3.4.38)

$$\langle G_{n,m+1}(t), P^*_{m+1}(D)g(t) \rangle = \langle P_{m+1}(D)G_{n,m+1}(t), g(t) \rangle$$

$$= \lim_{n_j \to \infty} n_j^{(m+1)/2} \langle P_{n_j,m}(G_{n,m+1},t)-G_{n,m+1}(t), g(t) \rangle$$

$$= \lim_{n_j \to \infty} n_j^{(m+1)/2} < P_{n_j,m}(G_{\eta,m+1}-G,t)-(G_{\eta,m+1}-G)(t),g(t) >$$

$$+ \lim_{n_j \to \infty} n_j^{(m+1)/2} < P_{n_j,m}(G,t)-G(t),g(t) >$$

i.e., $< G_{\eta,m+1}(t),P_{m+1}^*(D)g(t) > -$

$$- \lim_{n_j \to \infty} n_j^{(m+1)/2} < P_{n_j,m}(G,t)-G(t),g(t) >$$

$$= \lim_{n_j \to \infty} n_j^{(m+1)/2} < P_{n_j,m}(G_{\eta,m+1}-G,t)-(G_{\eta,m+1}-G)(t),g(t) >.$$

Applying Lemma 3.4.2 to the right hand side of the above expression we obtain

$$< G_{\eta,m+1}(t),P_{m+1}^*(D)g(t) > - \lim_{n_j \to \infty} n_j^{(m+1)/2} < P_{n_j,m}(G,t)-G(t),g(t)>$$

$$(3.4.39) \qquad \leq M_{14} \{||G_{\eta,m+1}^{(m)}-G^{(m)}||_{L_1(I)}+||G_{\eta,m+1}-G||_{L_1(I)}\} .$$

As before letting $\eta \to 0$ in (3.4.39)

$$(3.4.40) \quad < G(t),P_{m+1}^*(D)g(t) >$$

$$= \lim_{n_j \to \infty} n_j^{(m+1)/2} < P_{n_j,m}(G,t)-G(t),g(t) >.$$

After this we carry out the rest of the analysis as in the proof of Theorem 2.4.1 (saturation theorem for $P_n(f,k,t)$). The observation that $p_{m+1}(t) > 0$, $t \in (0,1)$, (see Corollary 3.1.2) completes the proof.

CHAPTER IV

L_p-APPROXIMATION BY LINEAR COMBINATIONS OF EXPONENTIAL TYPE OPERATORS

Approximation in C-norm by exponential type operators has been extensively studied (see Section 1.8). Here we show that under regularity conditions (1.5.8) and (1.5.9), exponential type operators also constitute an L_p-approximation process where $1 \leq p < \infty$. This is proved in Section 4.1. The linear combinations $S_n(.,k,t)$ of regular exponential type operators, as in C-norm case, give an improved order of convergence in L_p-norm for sufficiently smooth functions. This is in Section 4.2. In Sections 3 and 4 we prove inverse and saturation theorems for $\{S_n(.,k,t)\}$ in L_p-norm. The direct, inverse and saturation theorems are local in nature over contracting intervals. Some of the results obtained in this chapter will be useful in the next chapter as well.

In this and the next chapter we use the notations $I_j = [a_j, b_j]$ where $j = 1,2,3$, $A < a_j < a_{j+1}$ and $b_{j+1} < b_j < B$.

4.1 BASIC APPROXIMATION

We see from Lemma 1.8.4 that the operators $S_n(.,t)$ constitute a local approximation process with respect to C-norm for continuous functions satisfying certain growth conditions on (A,B). Here we prove that if, in addition, the operators

Proof. Let $\chi(u)$ be the characteristic function of I_1. Using Jensen's inequality

$$\int_{a_2}^{b_2} |S_n(f,t)|^p \, dt \le \int_{a_2}^{b_2} \int_A^B W(n,t,u)|f(u)|^p \, du \, dt$$

$$= \int_{a_2}^{b_2} \int_A^B \chi(u) \, W(n,t,u)|f(u)|^p \, du \, dt$$

$$+ \int_{a_2}^{b_2} \int_A^B (1-\chi(u))W(n,t,u)|f(u)|^p \, du \, dt$$

$$= J_1 + J_2, \text{ say.}$$

It follows from an application of Fubini's theorem and (1.5.8) that

$$J_1 \le a(n) \, ||f||^p_{L_p(I_1)}.$$

If $M_1 > 0$ is sufficiently large we can find a $c > 0$ such that

$$\frac{|u-t|^{2\ell}}{|u|^{2\ell+1}} > c, \text{ for all } |u| \ge M_1 \text{ and } t \in I_2.$$

Also, for those values of u which lie in $(-M_1, M_1) \setminus I_1$, we have $|u-t| > \delta$, where $\delta = \min\,(a_2-a_1, b_1-b_2)$.

Thus, by Fubini's theorem

$$J_2 = \int_A^B \int_{a_2}^{b_2} (1-\chi(u))W(n,t,u)|f(u)|^p \, dt \, du$$

$$\le \int_{|u| \ge M_1} \int_{a_2}^{b_2} (1-\chi(u))W(n,t,u)|f(u)|^p \frac{|u-t|^{2\ell p}}{c^p(1+|u|^{2\ell})^p} \, dt \, du$$

$$+ \int_{|u| < M_1} \int_{a_2}^{b_2} (1-\chi(u))W(n,t,u)|f(u)|^p \frac{|u-t|^{2\ell p}}{\delta^{2\ell p}} \, dt \, du.$$

We apply Corollaries 1.8.9 and 1.8.10 to obtain bounds for the first and second terms respectively. Thus

$$J_2 \leq \frac{M_2}{n^{\ell p}} ||f||^p_{L_p[A,B]} .$$

The estimates of J_1 and J_2 complete proof of the lemma.

Corollary 4.1.3. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then for any fixed positive number ℓ

$$(4.1.3) \quad ||S_n(f,k,t)||_{L_p(I_2)} \leq M \{ ||f||_{L_p(I_1)} + \frac{1}{n^\ell} ||f||_{L_p[A,B]} \},$$

where M is a constant, independent of f and n.

Proof. Since $S_n(f,k,t) = \sum_{j=0}^{k} c(j,k) S_{d_j n}(f,t)$, and $c(j,k)$'s do not depend on n the proof follows from Lemma 4.1.2.

Theorem 4.1.4. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then

$$(4.1.4) \qquad ||S_n(f,t)-f(t)||_{L_p(I_2)} = o(1), \quad (n \to \infty).$$

Proof. We choose a sequence $\{f_\sigma\}$ of continuous functions which have a compact support $\subset (A,B)$ and satisfy

$$(4.1.5) \qquad ||f_\sigma - f||_{L_p(I_1)} = o(1), \quad (\sigma \to \infty).$$

We have

$$||S_n(f,t)-f(t)||_{L_p(I_2)} \le ||S_n(f-f_\sigma,t)||_{L_p(I_2)}$$

$$+ ||S_n(f_\sigma,t)-f_\sigma(t)||_{L_p(I_2)} + ||f_\sigma(t)-f(t)||_{L_p(I_2)}.$$

We apply Lemma 4.1.2 to obtain an estimate for the first term on the right hand side of the above inequality. Thus, for a fixed $\epsilon > 0$, we have for sufficiently large values of n

$$||S_n(f,t)-f(t)||_{L_p(I_2)} \le \epsilon + ||f_\sigma(t)-f(t)||_{L_p(I_1)}$$

$$(4.1.6) \qquad + ||S_n(f_\sigma,t)-f_\sigma(t)||_{L_p(I_2)}$$

Finally, applying (4.1.5) and Lemma 1.8.4 to the second and third terms, respectively on the right hand side of the above inequality, we obtain (4.1.4).

Corollary 4.1.5. Let $1 \le p < \infty$ and $f \in L_p[A,B]$. Then

$$(4.1.7) \qquad ||S_n(f,t)-f(t)||_{L_p[A,B]} = o(1), \quad (n \to \infty).$$

Proof. By Lemma 4.1.1 we have $S_n f \in L_p[A,B]$ for every n. Now given $\epsilon > 0$ it follows from Lebesgue's dominated convergence theorem that there exist numbers a,b where $A < a < b < B$ such that

$$(4.1.8) \quad \|f(t)(1-\chi(t))\|_{L_p[A,B]} \le \epsilon$$

and $\|(S_n(f,t))(1-\chi(t))\|_{L_p[A,B]} \le \epsilon,$

where $\chi(t)$ is the characteristic function of $[a,b]$.
Thus (4.1.7) follows from Theorem 4.1.4 and estimates (4.1.8).

4.2 ERROR ESTIMATES

In this section we investigate, for sufficiently smooth function, the rapidity with which the linear combinations $S_n(\cdot,k,t)$ of regular exponential type operators converge in L_p-norm $(1 \le p < \infty)$. The first two results give error estimates in the approximation of sufficiently differentiable functions for the cases $1 < p < \infty$ and $p = 1$, respectively. Using these results, next we obtain a general error estimate in the L_p-norm in terms of $(2k+2)$th integral modulus of smoothness of the function.

Theorem 4.2.1. Let $1 < p < \infty$ and $f \in L_p[A,B]$. If f has 2k+2 derivatives on I_1 with $f^{(2k+1)} \in A.C.(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$, then for some constant M

$$(4.2.1) \quad \|S_n(f,k,t)-f(t)\|_{L_p(I_2)} \le \frac{M}{n^{k+1}} \{\|f^{(2k+2)}\|_{L_p(I_1)} + \|f\|_{L_p[A,B]}\}.$$

To prove the theorem we need the following proposition.

Proposition 4.2.2. Let $1 < p < \infty$, $h \in L_p[A,B]$ and $i,j \in \mathbb{N}^o$. Then for any fixed positive number ℓ

$$(4.2.2) \quad ||S_n(|u-t|^i|\int_t^u |u-w|^j|h(w)|dw|,t)||_{L_p(I_2)}$$

$$\leq M\{n^{-(i+j+1)/2}\, ||h||_{L_p(I_1)} + n^{-\ell}\, ||h||_{L_p[A,B]}\},$$

where M is a certain constant.

Furthermore, if $-\infty < A < B < +\infty$ then for some constant M_1

$$(4.2.3) \quad ||S_n(|u-t|^i|\int_t^u |u-w|^j|h(w)|dw|,t)||_{L_p[A,B]}$$

$$\leq M_1\, n^{-(i+j+1)/2}\, ||h||_{L_p[A,B]} .$$

Proof. Proceeding as in the proof of Proposition 2.2.2 we obtain (4.2.2) and (4.2.3). The only difference is that we have to use a bound of $\int_A^B W(n,t,u)|u-t|^s\, du$ in place of $\int_o^1 K(n,t,u)|u-t|^s\, du$ in the estimates of J_1 and J_2.

Proof of Theorem 4.2.1. For $t \in I_2$ and $u \in I_1$ there holds

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)\, dw.$$

Let $\chi(u)$ be the characteristic function of I_1. Then for $t \in I_2$

$$S_n(f,t)-f(t) = \int_A^B \chi(u)W(n,t,u)(f(u)-f(t))\, du$$

$$+ \int_A^B (1-\chi(u))\, W(n,t,u)(f(u)-f(t))\, du$$

$$(4.2.4) \qquad = J_1(t) + J_2(t), \text{ say.}$$

Using the expansion of f(u) about the point 't' in $J_1(t)$ we have

$$J_1(t) = \sum_{i=1}^{2k+1} \{ \frac{f^{(i)}(t)}{i!} \int_A^B \chi(u)\, W(n,t,u)(u-t)^i\, du\}$$

$$+ \frac{1}{(2k+1)!} \int_A^B \chi(u)\, W(n,t,u)\{ \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w)dw\}\, du$$

(4.2.5) $= J_{11}(t) + J_{12}(t)$, say.

It follows from Proposition 4.2.2 that

(4.2.6) $$||J_{12}||_{L_p(I_2)} \leq \frac{M_2}{n^{k+1}} ||f^{(2k+2)}||_{L_p(I_1)}.$$

To obtain L_p-norm estimate of the function $J_{11}(t)$ we rewrite

$$J_{11}(t) = (\sum_{i=1}^{2k+1} \frac{f^{(i)}(t)}{i!} \{\int_A^B W(n,t,u)(u-t)^i\, du\})$$

$$+ (\sum_{i=1}^{2k+1} \frac{f^{(i)}(t)}{i!} \{\int_A^B (\chi(u)-1)W(n,t,u)(u-t)^i\, du\})$$

(4.2.7) $= \Sigma_{11}(t) + \Sigma_{12}(t)$, say.

We shall first obtain L_p-bound for the function $\Sigma_{12}(t)$. Let $\delta = \min\,(a_2-a_1, b_1-b_2)$ and $s = 2(k+1)-i$. Then by Corollary 1.8.2 there holds for all $t \in I_2$

$$\int_A^B (\chi(u)-1)W(n,t,u)|u-t|^i du \leq \delta^{-s} \int_A^B W(n,t,u)|u-t|^{2(k+1)}\, du$$

(4.2.8) $$\leq M_3\, n^{-(k+1)}.$$

From this it follows that

$$||\Sigma_{12}||_{L_p(I_2)} \leq M_4\, n^{-(k+1)} \left(\sum_{i=1}^{2k+1} ||f^{(i)}||_{L_p(I_2)}\right).$$

Next we obtain L_p-bound for the function $J_2(t)$.

$$J_2(t) = \int_A^B (1-\chi(u))W(n,t,u)f(u)du - f(t)\int_A^B (1-\chi(u))W(n,t,u)du.$$

Applying Lemma 4.1.2 and (4.2.8) to first and second terms respectively

$$||J_2||_{L_p(I_2)} \leq M_5\, n^{-(k+1)}\, ||f||_{L_p[A,B]}.$$

Thus, we obtain from (4.2.4) to (4.2.8) and estimates of Σ_{12} and J_2 that

$$||S_n(f,t)-f(t) - \sum_{i=1}^{2k+1} \left(\frac{f^{(i)}(t)}{i!}\right) S_n((u-t)^i,t))||_{L_p(I_2)}$$

$$(4.2.9) \qquad = \left(\sum_{i=1}^{2k+2} ||f^{(i)}||_{L_p(I_1)} + ||f||_{L_p[A,B]}\right) O(n^{-(k+1)}),$$

$$(n \to \infty).$$

Hence, by Lemma 1.2.2 and (1.5.2)

$$||S_n(f,k,t)-f(t) - \sum_{i=1}^{2k+1} \left(\frac{f^{(i)}(t)}{i!}\right) S_n((u-t)^i,k,t))||_{L_p(I_2)}$$

$$= \left(||f^{(2k+2)}||_{L_p(I_1)} + ||f||_{L_p[A,B]}\right) O(n^{-(k+1)}), (n \to \infty).$$

This alongwith Lemmas 1.8.1 and 1.2.2 completes the proof.

Proceeding as in the proof of Theorem 4.2.1 and using second assertion (4.2.3) of Proposition 4.2.2 we obtain the following.

Corollary 4.2.3. Let $1 < p < \infty$ and $f \in L_p[A,B]$ where $A,B \in \mathbb{R}$. If f has $2k+2$ derivatives over $[A,B]$ with $f^{(2k+1)} \in A.C.[A,B]$ and $f^{(2k+2)} \in L_p[A,B]$, then for some constant M

$$(4.2.10)\quad ||S_n(f,k,t)-f(t)||_{L_p[A,B]}$$

$$\leq \frac{M}{n^{k+1}} \left(||f^{(2k+2)}||_{L_p[A,B]} + ||f||_{L_p[A,B]}\right).$$

Theorem 4.2.4. Let $f \in L_1[A,B]$. If f has $2k+1$ derivatives over I_1 with $f^{(2k)} \in A.C.(I_1)$ and $f^{(2k+1)} \in B.V.(I_1)$, then for some constant M

$$(4.2.11)\quad ||S_n(f,k,t)-f(t)||_{L_1(I_2)} \leq \frac{M}{n^{k+1}} \{||f^{(2k+1)}||_{B.V.(I_1)}$$

$$+ ||f^{(2k+1)}||_{L_1(I_2)} + ||f||_{L_1[A,B]}\}.$$

To prove this theorem we require the following proposition.

Proposition 4.2.5. Let $h \in B.V.(I_1)$ and $\chi(u)$ be the characteristic function of I_1. Then for $i,j \in \mathbb{N}^0$ there holds for some constant M_1

$$(4.2.12)\quad ||S_n(\chi(u)|u-t|^i|\int_t^u |u-w|^j\,|dh(w)||,t)||_{L_1(I_2)}$$

$$\leq \frac{M_1}{n^{(i+j+1)/2}} ||h||_{B.V.(I_1)}.$$

Furthermore, if $A,B \in \mathbb{R}$ and $h \in B.V.[A,B]$, then for some constant M_2

$$(4.2.13)\quad \left|\left| S_n\left(|u-t|^i \left|\int_t^u |u-w|^j |dh(w)|\right|, t\right)\right|\right|_{L_1[A,B]}$$

$$\leq \frac{M_2}{n^{(i+j+1)/2}} ||h||_{B.V.[A,B]} .$$

The proof follows in the manner of the proof of Proposition 2.2.5 where we use moment estimates of exponential type operators in place of those of the Bernstein-Kantorovitch polynomials.

Proof of Theorem 4.2.4. By Theorem 14.1 of [61] the given hypothesis on f implies that for each $u \in I_1$ and almost all $t \in I_2$

$$f(u) = \sum_{i=0}^{2k} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{(u-t)^{2k+1}}{(2k+1)!} f^{(2k+1)}(t)$$

$$+ \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} df^{(2k+1)}(w) .$$

We have for $t \in I_2$

$$S_n(f,k,t)-f(t) = S_n((f(u)-f(t))\,\chi(u),k,t)$$

$$+ S_n((1-\chi(u))(f(u)-f(t)),k,t)$$

$$(4.2.14)\qquad = J_1(t) + J_2(t), \text{ say.}$$

For almost all $t \in I_2$

$$J_1(t) = \sum_{i=1}^{2k} \left\{ \frac{f^{(i)}(t)}{i!} S_n((u-t)^i \chi(u),k,t) \right\}$$

$$+ \frac{f^{(2k+1)}(t)}{(2k+1)!} S_n((u-t)^{2k+1} \chi(u),k,t) +$$

$$+\frac{1}{(2k+1)!}\,S_n(\chi(u)\int_t^u (u-w)^{2k+1}\,df^{(2k+1)}(w)\quad ,k,t)$$

$$(4.2.15)\qquad = J_{11}(t) + J_{12}(t) + J_{13}(t),\ \text{say.}$$

Now, $$J_{11}(t) = \sum_{i=1}^{2k} \left\{ \frac{f^{(i)}(t)}{i!} \{S_n((u-t)^i,k,t) + S_n((\chi(u)-1)(u-t)^i,k,t)\}\right\}.$$

It follows from the estimates of Σ_{11} and Σ_{12} in Theorem 4.2.1 and an application of Lemma 1.2.2 that

$$(4.2.16)\quad ||J_{11}||_{L_1(I_2)} \leq \frac{M_1}{n^{k+1}}(||f^{(2k+1)}||_{L_1(I_2)}+||f||_{L_1(I_2)}).$$

Also, as in the estimate of J_2 in Theorem 4.2.1 we have

$$(4.2.17)\qquad |\,J_2||_{L_1(I_2)} \leq \frac{M_2}{n^{k+1}}\,||f||_{L_1[A,B]}.$$

By Proposition 4.2.5

$$(4.2.18)\qquad ||J_{13}||_{L_1(I_2)} \leq \frac{M_3}{n^{k+1}}\,||f^{(2k+1)}||_{B.V.(I_1)}.$$

As in the case of $J_{11}(t)$ we write

$$J_{12}(t) = \frac{f^{(2k+1)}(t)}{(2k+1)!}\{S_n((u-t)^{2k+1},k,t) + S_n((u-t)^{2k+1}(\chi(u)-1),k,t)\}.$$

Since $\sum_{j=0}^{k} c(j,k)\,d_j^{-m} = 0,\ m = 1,2,\dots,k$ the first term in $J_{12}(t)$ is estimated by applying Lemma 1.8.1 and using the fact that discarding a countable set does not

change L_1-norm. To obtain a L_1-bound for the second term we proceed as in the proof of estimate of Σ_{12} in Theorem 4.2.1. Finally

$$(4.2.19) \qquad ||J_{12}||_{L_1(I_2)} \leq \frac{M_4}{n^{k+1}} ||f^{(2k+1)}||_{L_1(I_2)} .$$

Combining (4.2.14) to (4.2.19) we complete proof of the theorem.

Corollary 4.2.6. Let $A,B \in \mathbb{R}$ and $f \in L_1[A,B]$. If f has $2k+1$ derivatives over $[A,B]$ with $f^{(2k)} \in A.C.[A,B]$ and $f^{(2k+1)} \in B.V.[A,B]$, then for some constant M

$$(4.2.20) \qquad ||S_n(f,k,t)-f(t)||_{L_1[A,B]} \leq \frac{M}{n^{k+1}} \{||f^{(2k+1)}||_{B.V.[A,B]} + ||f^{(2k+1)}||_{L_1[A,B]} + ||f||_{L_1[A,B]}\}.$$

Proceeding as in the proof of Theorem 4.2.4 and using (4.2.13) we obtain (4.2.20).

Theorem 4.2.7. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then, for sufficiently large values of n,

$$(4.2.21) \qquad ||S_n(f,k,t)-f(t)||_{L_p(I_2)} \leq M\{\omega_{2k+2}(f,n^{-1/2},p,I_1)+n^{-(k+1)}||f||_{L_p[A,B]}\},$$

where M is a constant independent of n and f.

Proof. Let $\chi(u)$ be the characteristic function of I_1. Writing $f\chi = \bar{f}$,

$$||S_n(f,k,t)-f(t)||_{L_p(I_2)} \le ||S_n(f-\bar{f},k,t)||_{L_p(I_2)}$$

$$+ ||S_n(\bar{f},k,t)-\bar{f}(t)||_{L_p(I_2)}.$$

Applying Lemma 4.1.2 to the first term on the right hand side of the above inequality we obtain

$$(4.2.22) \quad ||S_n(f,k,t)-f(t)||_{L_p(I_2)} \le M_1 n^{-(k+1)}$$

$$+ ||S_n(\bar{f},k,t)-\bar{f}(t)||_{L_p(I_2)}.$$

We choose numbers a^* and b^* such that $a_1 < a^* < a_2 < b_2 < b^* < b_1$. Let $\bar{f}_{n,2k+2}$ denote the Steklov mean corresponding to $\bar{f}$. So we have

$$||S_n(\bar{f},k,t)-\bar{f}(t)||_{L_p(I_2)}$$

$$\le ||S_n(\bar{f}-\bar{f}_{n,2k+2},k,t)||_{L_p(I_2)} + ||\bar{f}_{n,2k+2}(t)-\bar{f}(t)||_{L_p(I_2)}$$

$$+ ||S_n(\bar{f}_{n,2k+2},k,t)-\bar{f}_{n,2k+2}(t)||_{L_p(I_2)}.$$

An application of Lemma 4.1.2 to the first term on the right hand side gives

$$||S_n(\bar{f},k,t)-\bar{f}(t)||_{L_p(I_2)} \leq M_2 \{||\bar{f}-\bar{f}_{n,2k+2}||_{L_p[a^*,b^*]}$$

$$+ n^{-\ell}||\bar{f}-\bar{f}_{n,2k+2}||_{L_p[A,B]}$$

$$(4.2.23) \quad + ||S_n(\bar{f}_{n,2k+2},k,t)-\bar{f}_{n,2k+2}(t)||_{L_p(I_2)}\}.$$

Applying Theorems 4.2.1 and 4.2.4 and using the fact

$$||f^{(2k+2)}||_{L_1[a,b]} = ||f^{(2k+1)}||_{B.V.[a,b]},$$

we have

$$(4.2.24) \quad ||S_n(\bar{f}_{n,2k+2},k,t)-\bar{f}_{n,2k+2}(t)||_{L_p(I_2)}$$

$$\leq \frac{M_3}{n^{k+1}} (||\bar{f}^{(2k+2)}_{n,2k+2}||_{L_p[a^*,b^*]}+||\bar{f}_{n,2k+2}||_{L_p[A,B]}).$$

Combining (4.2.23) and (4.2.24) we obtain

$$||S_n(\bar{f},k,t)-\bar{f}(t)||_{L_p(I_2)}$$

$$\leq M_4\{||\bar{f}-\bar{f}_{n,2k+2}||_{L_p[a^*,b^*]}+n^{-(k+1)}||\bar{f}^{(2k+2)}_{n,2k+2}||_{L_p[a^*,b^*]}$$

$$(4.2.25) \quad +n^{-\ell}||\bar{f}-\bar{f}_{n,2k+2}||_{L_p[A,B]}+n^{-(k+1)}||\bar{f}_{n,2k+2}||_{L_p[A,B]}\}.$$

Using estimates (1.3.2), (1.3.3) and (1.3.4) and taking $\eta = n^{-1/2}$ and $\ell = k+1$, for all sufficiently large values of n, it follows from (4.2.25) that

$$\|S_n(\bar{f},k,t)-\bar{f}(t)\|_{L_p(I_2)} \leq M_5 \{\omega_{2k+2}(\bar{f},n^{-1/2},p,I_1)$$
$$+ n^{-(k+1)}\|f\|_{L_p(I_1)}\}.$$

Since $\bar{f} = f$ on I_1, (4.2.21) follows from (4.2.22) and the above inequality. This completes the proof of the theorem.

4.3 INVERSE THEOREM

We see from Theorem 4.2.7 that if $1 \leq p < \infty$, $f \in L_p[A,B]$ and $\omega_{2k+2}(f,\tau,p,I_1) = O(\tau^{\alpha})$, $(\tau \to 0)$, then

$$\|S_n(f,k,t)-f(t)\|_{L_p(I_2)} = O(n^{-\alpha/2}), \ (n \to \infty).$$

A corresponding local inverse theorem over contracting intervals for the sequence $\{S_n(.,k,t)\}$ of operators is as follows.

Theorem 4.3.1. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then for $0 < \alpha < 2k+2$

(4.3.1) $$\|S_n(f,k,t)-f(t)\|_{L_p(I_1)} = O(n^{-\alpha/2}), \ (n \to \infty),$$

implies that

(4.3.2) $$\omega_{2k+2}(f,\tau,p,I_2) = O(\tau^{\alpha}), \quad (\tau \to 0).$$

Remark. In the proof of this theorem without any loss of generality we can assume that the function f has a compact support contained in (A,B). For, let a and b be such that $A < a < a_1 < b_1 < b < B$. Let $\chi(u)$ be the characteristic function of $[a,b]$. Then

$$||S_n(f\chi,k,t)-(f\chi)(t)||_{L_p(I_1)} \leq ||S_n(f,k,t)-f(t)||_{L_p(I_1)} + ||S_n((1-\chi)f,k,t)||_{L_p(I_1)}.$$

and hence by (4.3.1) and Lemma 4.1.2

$$||S_n(f\chi,k,t)-(f\chi)(t)||_{L_p(I_1)} = O(n^{-\alpha/2}), \quad (n \to \infty).$$

Thus (4.3.1) is satisfied with f replaced by $f\chi$ and the latter has a compact support. This implies (4.3.2) as $f = \bar{f}$ on I_2.

In the proof of the theorem we shall require the following auxilliary results.

Lemma 4.3.2. Let $1 \leq p < \infty$ and $h \in L_p[A,B]$ have a compact support $\subset (A,B)$. Then for $i \in \mathbb{N}^0$ and for any fixed positive number ℓ

$$(4.3.3) \quad ||S_n(|u-t|^i|h(u)|,t)||_{L_p(I_2)} \leq M\{n^{-i/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p[A,B]}\},$$

where the constant M is independent of n and h.

Proof. Let $\chi(u)$ be the characteristic function of I_1. Using Jensen's inequality one has

$$\int_{a_2}^{b_2} |S_n(|u-t|^i|h(u)|,t)|^p\,dt \leq \int_{a_2}^{b_2}\int_A^B \chi(u)W(n,t,u)|u-t|^{ip}|h(u)|^p\,du\,dt +$$

$$+ \int_{a_2}^{b_2} \int_{A}^{B} (1-\chi(u))W(n,t,u)|u-t|^{ip}|h(u)|^p \, du \, dt$$

$$= J_1 + J_2, \text{say.}$$

By Fubini's theorem and Corollary 1.8.10

$$J_1 \leq M_1 \, n^{-ip/2} ||h||^p_{L_p(I_1)}.$$

Let $\delta = \min(a_2-a_1, b_1-b_2)$ and $s = (2\ell-i)p$. Then by Fubini's theorem

$$J_2 \leq \delta^{-s} \int_{A}^{B} \int_{a_2}^{b_2} (1-\chi(u))W(n,t,u)|u-t|^{2\ell p}|h(u)|^p \, dt \, du.$$

Since supp $h \subset (A,B)$ we have by Corollary 1.8.10

$$J_2 \leq M_2 \, n^{-\ell p} \, ||h||^p_{L_p[A,B]}.$$

The lemma follows from estimates of J_1 and J_2.

Lemma 4.3.3. Let $1 \leq p < \infty$ and $h \in L_p[A,B]$ with supp $h \subset [a,b]$, where $A < a < b < B$. Then

$$(4.3.4) \qquad ||S_n^{(2k+2)}(h,t)||_{L_p[a,b]} \leq M \, n^{k+1} ||h||_{L_p[a,b]}.$$

In addition if h has $2k+2$ derivatives with $h^{(2k+1)} \in A.C.[a,b]$ and $h^{(2k+2)} \in L_p[a,b]$ then

$$(4.3.5) \quad ||S_n^{(2k+2)}(h,t)||_{L_p[a,b]} \leq M_1 ||h^{(2k+2)}||_{L_p[a,b]},$$

the constants M, M_1 are independent of n and h.

Proof. By Lemma 1.8.3 for $t \in [a,b]$,

$$S_n^{(2k+2)}(h,t) = \sum_{i,j} \{ n^{i+j} \frac{a_{ij}^{(2k+2)}(t)}{(p(t))^{2k+2}} \int_A^B W(n,t,u)(u-t)^j h(u)du \},$$

where $i,j \in \mathbb{N}^0$ satisfy $2i+j \leq 2k+2$.

As $p(t)$ is bounded over $[a,b]$, Lemma 4.3.2 implies that

$$||S_n^{(2k+2)}(h,t)||_{L_p[a,b]} \leq M \ n^{k+1}||h||_{L_p[a,b]}.$$

To prove (4.3.5), since for $u,\ t \in [a,b]$

$$h(u) = \sum_{i=0}^{2k+1} \frac{(u-t)^i}{i!} h^{(i)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} h^{(2k+2)}(w)\,dw,$$

we have

$$S_n(h,x) = \sum_{i=0}^{2k+1} \{ \frac{h^{(i)}(t)}{i!} S_n((u-t)^i,x) \}$$

$$+ \frac{1}{(2k+1)!} S_n(\int_t^u (u-w)^{2k+1} h^{(2k+2)}(w)dw,\ x)$$

As $S_n(\cdot,x)$ maps algebraic polynomials into algebraic polynomials of same degree (see Lemma 1.8.1) we have

$$S_n^{(2k+2)}(h,t) = \frac{1}{(2k+1)!} S_n^{(2k+2)}(\int_t^u (u-w)^{2k+1} h^{(2k+2)}(w)dw, t)$$

$$= \frac{1}{(2k+1)!} \{ \sum_{i,j} n^{i+j} \frac{a_{ij}^{(2k+2)}(t)}{(p(t))^{2k+2}} \{ \int_A^B W(n,t,u)(u-t)^j \times$$

$$\times \{ \int_t^u (u-w)^{2k+1} h^{(2k+2)}(w)dw \} du \}\}.$$

Hence it follows from Propositions 4.2.2 and 4.2.5 that

$$||S_n^{(2k+2)}(h,t)||_{L_p[a,b]} \le M_1||h^{(2k+2)}||_{L_p[a,b]},$$

completing the proof.

Proof of Theorem 4.3.1. Let x_i, y_i, $i = 1,2,3,4$, be pairs of points such that $a_1 < x_i < a_2$, $b_2 < y_i < b_1$, $x_i < x_{i+1}$ and $y_{i+1} < y_i$. We choose a function $g \in C_0^{2k+2}$ such that $\operatorname{supp} g \subset (x_3, y_3)$ and $g(t) = 1$ for $t \in [x_4, y_4]$.

Writing $fg = \bar{f}$, for all values of $\gamma \le \tau$, as in the proof of Theorem 2.3.1, we have

$$||\Delta_\gamma^{2k+2}\,\bar{f}(t)||_{L_p[x_3,y_3]} \le ||\Delta_\gamma^{2k+2}\{\bar{f}(t)-S_n(\bar{f},k,t)\}||_{L_p[x_3,y_3]}$$

$$+\ \gamma^{2k+2}\{||S_n^{(2k+2)}(\bar{f}-\bar{f}_{\eta,2k+2},k,t)||_{L_p[x_3',y_3']}$$

$$+\ ||S_n^{(2k+2)}(\bar{f}_{\eta,2k+2},k,t)||_{L_p[x_3',y_3']}\},$$

where $x_3' = x_3$ and $y_3' = y_3+(2k+2)\gamma$.

Applying Lemma 4.3.3, for small values of η

$$||\Delta_\gamma^{2k+2}\,\bar{f}(t)||_{L_p[x_3,y_3]} \le ||\Delta_\gamma^{2k+2}\{\bar{f}(t)-S_n(\bar{f},k,t)\}||_{L_p[x_3,y_3]}$$

$$+\ \gamma^{2k+2}M_1\{n^{k+1}||\bar{f}-\bar{f}_{\eta,2k+2}||_{L_p[x_3,y_3]}$$

$$+\ ||\bar{f}_{\eta,2k+2}^{(2k+2)}||_{L_p[x_3,y_3]}\}.$$

This, in conjunction with estimates (1.3.3) and (1.3.2), gives

$$(4.3.6)\quad ||\Delta_\gamma^{2k+2}\bar{f}(t)||_{L_p[x_3,y_3]}$$

$$\leq ||\Delta_\gamma^{2k+2}\{\bar{f}(t)-S_n(\bar{f},k,t)\}||_{L_p[x_3,y_3]}$$

$$+M_1'\gamma^{2k+2}(n^{k+1}+n^{-(2k+2)})\omega_{2k+2}(\bar{f},n,p,[x_2,y_2]).$$

To complete the proof of the theorem we are now left to show that

$$(4.3.7)\quad ||\Delta_\gamma^{2k+2}\{\bar{f}(t)-S_n(\bar{f},k,t)\}||_{L_p[x_3,y_3]}=O(n^{-\alpha/2}),\ (n\to\infty).$$

As usual we prove it by an induction on α.

Consider first the case when $\alpha\leq 1$. By (4.3.1)

$$||S_n(fg,k,t)-(fg)(t)|_{L_p[x_3,y_3]}$$

$$\leq ||S_n((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]}$$

$$+||S_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$\leq M_2n^{-\alpha/2}+\sum_{j=0}^{k}\{|c(j,k)|\ ||S_{d_jn}(f(u)(u-t)g'(\xi),t)||_{L_p[x_3,y_3]}\},$$

for some ξ lying between u and t.

Applying Lemma 4.3.2, we have

$$||S_n(fg,k,t)-(fg)(t)||_{L_p[x_3,y_3]}\leq M_2'\,n^{-\alpha/2},$$

which proves (4.3.7).

Now assuming that for some $r \leq 2k+1$, the theorem holds for all values of α satisfying $r-1 \leq \alpha < r$, we prove that the theorem holds good for all α satisfying $r \leq \alpha < r+1$.

We have

$$||S_n(fg,k,t)-(fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq ||S_n((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||S_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$\leq \frac{M_3}{n^{\alpha/2}} + ||S_n((f(u)-f_{n,k+2}(u))(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||S_n((f_{n,2k+2}(u)-f_{n,2k+2}(t))(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||S_n(f_{n,2k+2}(t)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

(4.3.8) $\qquad = \frac{M_3}{n^{\alpha/2}} + J_1 + J_2 + J_3$, say.

By (1.3.4) and Lemma 1.8.5

(4.3.9) $\qquad J_3 \leq \frac{M_3'}{n^{k+1}}$.

First applying the mean value theorem and then Lemma 4.3.2 we get

$$J_1 = ||S_n((f(u)-f_{n,2k+2}(u))(u-t)g'(\xi),k,t)||_{L_p[x_3,y_3]}$$

$$\leq ||g'||_{C(I_1)} \{ \sum_{j=0}^{k} |c(j,k)| \times$$

$$\times ||S_{d_j n}(|f(u)-f_{n,2k+2}(u)|.|u-t|,t)||_{L_p[x_3,y_3]} \}$$

$$\leq M_4 \{ n^{-1/2} ||f-f_{n,2k+2}||_{L_p[x_2,y_2]}$$

$$+ n^{-(k+1)} ||f-f_{n,2k+2}||_{L_p[A,B]} \}.$$

Using estimates (1.3.3) and (1.3.4) this is further estimated as

$$(4.3.10) \quad J_1 \leq M_4' \{ n^{-1/2} \omega_{2k+2}(f,\eta,p,[x_1,y_1])$$

$$+ n^{-(k+1)} ||f||_{L_p[A,B]} \}.$$

We estimate J_2 as follows. For some ξ lying between u and t

$$(f_{n,2k+2}(u)-f_{n,2k+2}(t))(g(u)-g(t))$$

$$= \{ \sum_{i=1}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}_{n,2k+2}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f^{(2k+2)}_{n,2k+2}(w)dw \} \times$$

$$\times \{ \sum_{i=1}^{2k} \frac{(u-t)^i}{i!} g^{(i)}(t) + \frac{(u-t)^{2k+1}}{(2k+1)!} g^{(2k+1)}(\xi) \} .$$

Therefore, $J_2 \leq$

$$\{ \sum_{i=1}^{2k+1} \sum_{j=1}^{2k} \frac{1}{i!\,j!} ||S_n(f^{(i)}_{n,2k+2}(t)g^{(j)}(t)(u-t)^{i+j},k,t)||_{L_p[x_3,y_3]} \}+$$

$$+ \frac{1}{(2k+1)!} \{ \sum_{i=1}^{2k+1} \frac{1}{i!} \times$$

$$\times ||S_n(f_{n,2k+2}^{(i)}(t)g^{(2k+1)}(\xi)(u-t)^{2k+i+1},k,t)||_{L_p[x_3,y_3]} \}$$

$$+ \{ \sum_{i=1}^{2k} \frac{1}{(2k+1)!\,i!} \times$$

$$\times ||g^{(i)}(t)S_n((u-t)^i \{ \int_t^u (u-w)^{2k+1} f_{n,2k+2}^{(2k+2)}(w)dw \},k,t)||_{L_p[x_3,y_3]} \}$$

$$+ \frac{1}{((2k+1)!)^2} \times$$

$$\times ||S_n((u-t)^{2k+1}g^{(2k+1)}(\xi)\int_t^u (u-w)^{2k+1} f_{n,2k+2}^{(2k+2)}(w)dw,k,t)||_{L_p[x_3,y_3]}$$

$$(4.3.11) \qquad = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \text{ say.}$$

By Propositions 4.2.2 and 4.2.5, for any fixed positive number ℓ,

$$\Sigma_3 \leq M_5 \{ (\sum_{i=1}^{2k} \frac{1}{n^{k+1+i/2}}) ||f_{n,2k+2}^{(2k+2)}||_{L_p[x_2,y_2]}$$

$$+ \frac{1}{n^\ell} ||f_{n,2k+2}^{(2k+2)}||_{L_p[A,B]} \},$$

and

$$\Sigma_4 \leq M_5 \{ \frac{1}{n^{2k+3/2}} ||f_{n,2k+2}^{(2k+2)}||_{L_p[x_2,y_2]}$$

$$+ \frac{1}{n^\ell} ||f_{n,2k+2}^{(2k+2)}||_{L_p[A,B]} \}.$$

Using estimates (1.3.2) and (1.3.5)

(4.3.12) $$\Sigma_3 \leq M_5' \{ \frac{1}{n^{2k+2}} \frac{1}{n^{k+1+1/2}} \omega_{2k+2}(f,n,p,[x_1,y_1]) + \frac{1}{n^{2k+2}} \frac{1}{n^{\ell}} ||f||_{L_p[A,B]} \},$$

and

(4.3.13) $$\Sigma_4 \leq M_5' \{ \frac{1}{n^{2k+2}} \frac{1}{n^{2k+3/2}} \omega_{2k+2}(f,n,p,[x_1,y_1]) + \frac{1}{n^{2k+2}} \frac{1}{n^{\ell}} ||f||_{L_p[A,B]} \}.$$

It follows from Lemma 1.8.1 and the fact

$$\sum_{j=0}^{k} c(j,k)\, d_j^{-m} = 0, \; m = 1,2,\ldots,k, \text{ that}$$

$$\Sigma_1 \leq \frac{M_6}{n^{k+1}} \left(\sum_{i=1}^{2k+1} ||f^{(i)}_{n,2k+2}||_{L_p[x_3,y_3]} \right).$$

It follows from Corollary 1.8.2 that

$$\Sigma_2 \leq \frac{M_6'}{n^{k+1}} \left(\sum_{i=1}^{2k+1} ||f^{(i)}_{n,2k+2}||_{L_p[x_3,y_3]} \right).$$

Using Lemma 1.2.2 and then applying estimates (1.3.2) and (1.3.4) to the right side of the above two inequalities we obtain

(4.3.14) $$\Sigma_1 \leq \frac{M_7}{n^{k+1}} \{ \frac{1}{n^{2k+1}} \omega_{2k+1}(f,n,p,[x_2,y_2]) + ||f||_{L_p[A,B]} \},$$

and

(4.3.15) $$\Sigma_2 \leq \frac{M_7'}{n^{k+1}} \{ \frac{1}{n^{2k+1}} \omega_{2k+1}(f,n,p,[x_2,y_2]) + ||f||_{L_p[A,B]} \}.$$

By the induction hypothesis we can assume that

$$(4.3.16)\quad \omega_{2k+2}(f,\eta,p,[x_1,y_1]) = O(\eta^{\alpha-1}), \quad (\eta \to 0).$$

By Corollary 1.3.4 this implies that

$$(4.3.17)\quad \omega_{2k+1}(f,\eta,p,[x_1,y_1]) = O(\eta^{\alpha-1}), \quad (\eta \to 0).$$

Therefore, by taking $\eta = n^{-1/2}$ and $\ell = 2k+2$ inequalities (4.3.12) to (4.3.15) and estimates (4.3.16) and (4.3.17) imply that

$$\Sigma_1,\ \Sigma_2,\ \Sigma_3 \text{ and } \Sigma_4 \le \frac{M_8}{n^{\alpha/2}}\ .$$

Putting these estimates of Σ_k in (4.3.11) we conclude that

$$(4.3.18)\qquad J_2 \le \frac{M_8'}{n^{\alpha/2}}\ .$$

From (4.3.10) and (4.3.16) we obtain

$$(4.3.19)\qquad J_1 \le \frac{M_9}{n^{\alpha/2}}\ .$$

From the bounds for J_1, J_2 and J_3 we finally obtain from (4.3.8)

$$||S_n(fg,k,t)-(fg)(t)||_{L_p[x_3,y_3]} = O(n^{-\alpha/2}), \quad (n \to \infty).$$

This proves (4.3.7).

4.4 SATURATION THEOREM

In this section we show that as in C-norm (see Lemma 1.8.7) the linear combinations $S_n(.,k,t)$ of regular exponential type operators are saturated in L_p-norm $(1 \le p < \infty)$

with the order $O(n^{-(k+1)})$. Furthermore, we see from Theorem 4.4.1 (proved in this section) and Lemma 1.8.7 that the larger the p the smaller is the saturation class. But the trivial class is essentially the same for all p ($1 \leq p \leq \infty$).

Theorem 4.4.1. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then, in the following, the implications "(i) ==> (ii) ==> (iii)" and "(iv) ==> (v) ==> (vi)" hold.

(i) $\quad ||S_n(f,k,t)-f(t)||_{L_p(I_1)} = O(n^{-(k+1)}), \quad (n \to \infty)$;

(ii) f coincides a.e. with a function F on I_2 having 2k+2 derivatives such that (a) when $p > 1$, $F^{(2k+1)} \in A.C.(I_2)$ and $F^{(2k+2)} \in L_p(I_2)$, and (b) when $p = 1$, $F^{(2k)} \in A.C.(I_2)$ and $F^{(2k+1)} \in B.V.(I_2)$;

(iii) $\quad ||S_n(f,k,t)-f(t)||_{L_p(I_3)} = O(n^{-(k+1)}), \quad (n \to \infty)$;

(iv) $\quad ||S_n(f,k,t)-f(t)||_{L_p(I_1)} = o(n^{-(k+1)}), \quad (n \to \infty)$;

(v) f coincides a.e. with a function F on I_2, where F is 2k+2 times continuously differentiable on I_2 and satisfies

$$\sum_{j=k+1}^{2k+2} Q(j,k,t)F^{(j)}(t) = 0, \quad t \in I_2,$$

where $Q(j,k,t)$ are the polynomials occuring in Lemma 1.8.5;

(vi) $\quad ||S_n(f,k,t)-f(t)||_{L_p(I_3)} = o(n^{-(k+1)}), \quad (n \to \infty)$.

Proof. We see from the proof of saturation theorem of Chapter II (Chapter III) that crux of the proof is Lemma 2.4.1 (Lemma 3.4.1). Here we prove a similar lemma.

Lemma 4.4.2. Let $1 \leq p < \infty$, $h \in L_p[A,B]$ with $\text{supp } h \subset I_1^o$ and $g \in C_o^{2k+2}$ with $\text{supp } g \subset I_1^o$. Then

$$(4.4.1) \qquad |\langle S_n(h,k,t)-h(t),g(t)\rangle| \leq \frac{M}{n^{k+1}} ||h||_{L_1(I_1)},$$

where M is a constant independent of h and n.

Proof of Lemma. By definition

$$\langle S_n(h,k,t)-h(t),g(t)\rangle = \sum_{j=0}^{k} \{c(j,k) \langle S_{d_j n}(h,t)-h(t),g(t)\rangle\}$$

$$= \sum_{j=0}^{k} \{c(j,k) \langle h(u),S^*_{d_j n}(g,u)-g(u)\rangle\}$$

$$= \langle h(u), S_n^*(g,k,u)-g(u)\rangle.$$

Hence applying Lemma 1.8.11 to the right hand side of the above identity we obtain

$$|\langle S_n(h,k,t)-h(t),g(t)\rangle| \leq \frac{M}{n^{k+1}} ||h||_{L_1(I_1)}.$$

Now the rest of the proof of this theorem goes along the lines of the proof of Theorem 2.4.1. Hence we omit the details.

CHAPTER V

L_p-APPROXIMATION BY INTERPOLATORY MODIFICATIONS OF EXPONENTIAL TYPE OPERATORS

In this chapter L_p-approximation by interpolatory modifications $S_{n,m}(\cdot,t)$ of regular exponential type operators (Section 6, Chapter I) is studied. Unlike the case of Bernstein-Kantorovitch polynomials, here we have to divide the kernel $W(n,t,u)$ by $(|u-t|^{m_o+2}+1)$, $m \leq m_o$ to make the operators $S_{n,m}(\cdot,t)$ L_p-bounded. This is shown in Section 1. In Section 2 we obtain error estimates in L_p-norm $(1 \leq p < \infty)$ in terms of derivatives of the function and also in terms of (m+1)th integral modulus of smoothness of the function. In Sections 3 and 4 we prove inverse and saturation theorems. As before, the results are in a local set-up over contracting intervals.

5.1 BASIC APPROXIMATION

In this section we first obtain a formula expressing moments of modified exponential type operators $S_{n,m}(\cdot,t)$ in terms of moments of exponential type operators $S_n(\cdot,t)$. Next, we prove that the operators $S_{n,m}(\cdot,t)$ are L_p-bounded over compact subsets of (A,B). Using this we finally prove that they constitute an L_p-approximation method.

Lemma 5.1.1. Let K be a compact subset of (A,B). Then, for $k \in \mathbb{N}$ there hold

(i) If $k \leq m$, $S_{n,m}((u-t)^k, t) = 0$;

(ii) $S_{n,m}((u-t)^{m+1}, t)$

$$= (-1)^m S_n\left(\prod_{i=0}^{m} \left(u-t+\frac{i}{n^{1/2}}\right), t\right) + O\left(\frac{1}{n^{m+3/2}}\right), \quad (n \to \infty),$$

uniformly in $t \in K$.

(iii) If $k > m+1$,

$$S_{n,m}((u-t)^k, t) = \sum_{r=0}^{k-1} \frac{a_r}{n^{r/2}} S_n((u-t)^{k-r}, t) + O\left(\frac{1}{n^{(m+k+2)/2}}\right),$$

as $n \to \infty$, uniformly in $t \in K$, where a_r's are certain real numbers.

Proof. We have from (1.6.2)

$$S_{n,m}((u-t)^k, t)$$

$$(5.1.1) \quad = \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} \times$$

$$\times \left\{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} \left(\prod_{i=0}^{j-1} \left(t-u-\frac{i}{n^{1/2}}\right)\right)(\Delta^j (u-t)^k)\right\} du,$$

where product $\prod_{i=0}^{j-1} \left(t-u-\frac{i}{n^{1/2}}\right)$ for $j = 0$ is interpreted as 1 and $m \leq m_o$.

From (5.1.1) and (3.1.3), (i) of Lemma 5.1.1 follows. Also, as in Lemma 3.1.1,

$$S_{n,m}((u-t)^{m+1}, t) = (-1)^m \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} \left(\prod_{i=0}^{m} \left(u-t+\frac{i}{n^{1/2}}\right)\right) du$$

$$= (-1)^m \int_A^B W(n,t,u) \left(\prod_{i=0}^{m} \left(u-t+\frac{i}{n^{1/2}}\right)\right) du$$

$$- (-1)^m \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} \left(\prod_{i=0}^{m} (u-t+\frac{i}{n^{1/2}}) \right) du$$

(5.1.2) $\quad = J_1(t) - J_2(t)$, say.

As $|J_2(t)| \le \{ \sum_{r=0}^{m} \frac{b_r}{n^{r/2}} \int_A^B W(n,t,u)|u-t|^{m+m_o+3-r} du \}$,

where b_r's are positive numbers; applying Corollary 1.8.2

(5.1.3) $\quad |J_2(t)| \le \frac{M}{n^{(m+m_o+3)/2}} \le \frac{M}{n^{m+3/2}}$, $t \in K$.

Thus (ii) follows from (5.1.2) and (5.1.3).

(iii) Proceeding as in the proof of (iii) of Lemma 3.1.1, we have

$$\sum_{j=0}^{m} \{ \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}}))(\Delta^j (u-t)^k) \} = \sum_{r=0}^{k-1} \frac{a_r}{n^{r/2}} (u-t)^{k-r},$$

where a_r are certain real numbers. Hence by (5.1.1)

$$S_{n,m}((u-t)^k,t) = \sum_{r=0}^{k-1} \{ \frac{a_r}{n^{r/2}} S_n(\frac{(u-t)^{k-r}}{1+|u-t|^{m_o+2}},t) \}$$

$$= \sum_{r=0}^{k-1} \frac{a_r}{n^{r/2}} \{ S_n((u-t)^{k-r},t)$$

$$- S_n(\frac{(u-t)^{k-r}\,|u-t|^{m_o+2}}{1+|u-t|^{m_o+2}},t) \}$$

(5.1.4) $$= \sum_{r=0}^{k-1} \frac{a_r}{n^{r/2}} (J_1(t) - J_2(t)), \text{ say.}$$

Proceeding as in the estimate of $J_2(t)$ in (ii)

$$(5.1.5)\quad |J_2(t)| \le \frac{M_1}{n^{(m_0+k+2-r)/2}} \le \frac{M_1}{n^{(m+k+2-r)/2}}\,,\quad t \in K.$$

(5.1.4) and (5.1.5) complete the proof of (iii).

Corollary 5.1.2. There holds

$$(5.1.6)\qquad S_{n,m}((u-t)^{m+1},t) = (-1)^m \frac{q_{m+1}(t)}{n^{(m+1)/2}} + o\left(\frac{1}{n^{(m+1)/2}}\right), (n \to \infty),$$

where $q_{m+1}(t)$ is a polynomial in t of degree $\le m+1$ and $q_{m+1}(t) > 0$ on (A,B). The o-term holds uniformly with respect to $t \in K$.

Proof. Proceeding as in the proof of Corollary 3.1.2 and using Lemma 1.8.1 we obtain (5.1.6).

Theorem 5.1.3. Let $1 \le p < \infty$ and $f \in L_p[A,B]$. Then for any positive number ℓ and sufficiently large values of n

$$(5.1.7)\quad ||S_{n,m}(f,t)||_{L_p(I_2)} \le M\{||f||_{L_p(I_1)} + n^{-\ell}||f||_{L_p[A,B]}\},$$

M being a constant independent of n and f.

Proof. A typical component of $S_{n,m}(f,t)$

$$= \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_0+2}} \left\{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} \left(\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})\right) \Delta^j f(u)\right\} du$$

is of the type

$$c \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_0+2}} (n^{1/2}(t-u))^{j-r}\, \Delta^j f(u)du = T_1(t), \text{ say,}$$

where $T_1(t) = T_1(t;j,r)$, $0 \le j \le m$, $0 \le r \le j-1$ and c is a scalar (when $j = 0$, r takes value '0' only). Let $\chi(u)$ be characteristic function of $[a^*,b^*]$ where $a_1 < a^* < a_2 < b_2 < b^* < b_1$.

Using Jensen's inequality

$$\int_{a_2}^{b_2} |T_1(t)|^p \, dt$$

$$\le |c|^p \int_{a_2}^{b_2} \int_A^B \frac{W(n,t,u)}{(1+|u-t|^{m_o+2})^p} (n^{1/2}|t-u|)^{(j-r)p} |\Delta^j f(u)|^p \, du \, dt$$

$$= |c|^p \, n^{(j-r)p/2} \{ \int_{a_2}^{b_2} \int_A^B \frac{\chi(u) W(n,t,u)}{(1+|u-t|^{m_o+2})^p} |t-u|^{(j-r)p} |\Delta^j f(u)|^p \, du \, dt$$

$$+ \int_{a_2}^{b_2} \int_A^B \frac{(1-\chi(u)) W(n,t,u)}{(1+|u-t|^{m_o+2})^p} |t-u|^{(j-r)p} |\Delta^j f(u)|^p \, du \, dt \}$$

$$= J_1 + J_2, \text{ say.}$$

Applying Fubini's theorem and Corollary 1.8.10 we have for large values of n

(5.1.8) $$J_1 \le M_1 ||f||^p_{L_1(I_1)}.$$

To estimate J_2, we note that for some $M_o > 0$ there exists a c_o such that for all $t \in I_2$ and $|u| \ge M_o$,

$$|u-t| > c_o |u|.$$

Also, for those values of u which lie in $(-M_o, M_o) \setminus [a^*,b^*]$, we have, with $\delta = \min(a_2-a^*, b^*-b_2)$, $|u-t| \ge \delta$.

Hence

$$J_2 = |c|^p \, n^{(j-r)p/2} \int_{a_2}^{b_2} \{ (\int_{|u| < M_o} + \int_{|u| \geq M_o}) \frac{(1-\chi(u))W(n,t,u)}{(1+|u-t|^{m_o+2})^p} \times$$

$$\times |t-u|^{(j-r)p} |\Delta^j f(u)|^p \, du \} \, dt$$

$$\leq |c|^p \, n^{(j-r)p/2} \int_{a_2}^{b_2} \int_{|u| < M_o} \frac{(1-\chi(u))W(n,t,u)}{\delta^{2\ell p}(1+|u-t|^{m_o+2})^p} \times$$

$$\times |t-u|^{(j+2\ell-r)p} |\Delta^j f(u)|^p \, du \, dt$$

$$(5.1.9) \qquad + |c|^p \, n^{(j-r)p/2} \int_{a_2}^{b_2} \int_{|u| \geq M_o} \frac{(1-\chi(u)) \, W(n,t,u)}{(c_o|u|)^{(m_o+2\ell+2)p}} \times$$

$$\times |t-u|^{(j+2\ell-r)p} |\Delta^j f(u)|^p \, du \, dt$$

Using Fubini's theorem to interchange the integrals in u and t we obtain from (5.1.9)

$$J_2 \leq M_2 \, n^{(j-r)p/2} \; \{ \int_{|u| < M_o} \int_{a_2}^{b_2} W(n,t,u)|t-u|^{(j+2\ell-r)p} |\Delta^j f(u)|^p \, dt \, du$$

$$+ \int_{|u| \geq M_o} \int_{a_2}^{b_2} \frac{W(n,t,u)}{u^{(m_o+2\ell+2)p}} |t-u|^{(j+2\ell-r)p} |\Delta^j f(u)|^p \, dt \, du \}$$

$$(5.1.10) \qquad = J_{21} + J_{22}, \text{ say.}$$

Applying Corollary 1.8.10 we obtain

$$(5.1.11) \qquad J_{21} \leq M_3 \, n^{-\ell p} \, \| f \|^p_{L_p[A,B]} .$$

J_{22} is estimated as follows. Let $s > (j+2\ell-r)p$ be an even integer. Then, writing $\theta = (j+2\ell-r)p/s$, from Holder's inequality and (1.5.8)

$$\int_{a_2}^{b_2} W(n,t,u)|u-t|^{(j+2\ell-r)p}\,dt$$
$$\leq \left(\int_{a_2}^{b_2} W(n,t,u)(u-t)^s\,dt\right)^\theta \left(\int_{a_2}^{b_2} W(n,t,u)dt\right)^{1-\theta}$$
$$\leq (a(n))^{1-\theta} \left(\int_A^B W(n,t,u)(u-t)^s\,dt\right)^\theta.$$

By Corollary 1.8.9, $\int_A^B W(n,t,u)(u-t)^s\,dt$ is a polynomial in u of degree $\leq$ s; hence

$\frac{1}{|u|^{(m_o+2\ell+2)p}} \left(\int_A^B W(n,t,u)(u-t)^s\,dt\right)^\theta$, as a function of u, is bounded. Moreover, Corollary 1.8.9 implies that

$$\frac{1}{|u|^{(m_o+2\ell+2)p}} \left(\int_A^B W(n,t,u)(u-t)^s\,dt\right)^\theta \leq \frac{M_4}{n^{(j+2\ell-r)p/2}}.$$

Therefore,

$$J_{22} \leq M_4' \, n^{-\ell p}\left(\int_{|u|\geq M_o} |\Delta^j f(u)|^p\,du\right)$$
$$(5.1.12) \qquad \leq M_5 \, n^{-\ell p} \, ||f||^p_{L_p[A,B]}.$$

Collecting (5.1.8) to (5.1.12) we see that

$$||T_1||_{L_p(I_2)} \leq M_5' \left(||f||_{L_p(I_1)} + n^{-\ell}\,||f||_{L_p[A,B]}\right).$$

Since $T_1(t)$ is a typical component of $S_{n,m}(f,t)$, the theorem follows from the above L_p-estimate of $T_1(t)$.

Corollary 5.1.4. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then for some constant M

$$(5.1.13) \qquad ||S_{n,m}(f,t)||_{L_p[A,B]} \leq M||f||_{L_p[A,B]}.$$

Proof. Proceeding as in the above theorem with the characteristic function of $[a^*,b^*]$ replaced by characteristic function of $[A,B]$, we prove (5.1.13).

Theorem 5.1.5. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then

$$(5.1.14) \qquad ||f(t)-S_{n,m}(f,t)||_{L_p(I_2)} = o(1), \quad (n \to \infty).$$

Proof. We choose a sequence $\{f_\sigma\}$ of continuous functions having a compact support $\subset (A,B)$ such that

$$(5.1.15) \qquad ||f_\sigma(t)-f(t)||_{L_p(I_1)} = o(1), \quad (\sigma \to \infty).$$

Then,

$$||S_{n,m}(f,t)-f(t)||_{L_p(I_2)}$$

$$\leq ||S_{n,m}(f-f_\sigma,t)||_{L_p(I_2)} + ||f_\sigma(t)-f(t)||_{L_p(I_2)}$$

$$+ ||S_{n,m}(f_\sigma,t)-f_\sigma(t)||_{L_p(I_2)}$$

$$(5.1.16) \qquad = ||J_1(t)||_{L_p(I_2)}+||J_2(t)||_{L_p(I_2)}+||J_3(t)||_{L_p(I_2)}, \text{say.}$$

By Theorem 5.1.3 we have for any $\ell > 0$

$$||J_1(t)||_{L_p(I_2)} \leq M_1\{||f-f_\sigma||_{L_p(I_1)} + n^{-\ell}\,||f-f_\sigma||_{L_p[A,B]}\}.$$

Then for a given $\epsilon > 0$ it follows from (5.1.15) that for large values of n and σ

$$(5.1.17) \qquad ||J_1(t)||_{L_p(I_2)} \leq \epsilon$$

and

$$(5.1.18) \qquad ||J_2(t)||_{L_p(I_2)} \leq \epsilon.$$

Therefore, to complete the proof of theorem we have to show now that

$$||J_3(t)||_{L_p(I_2)} \leq \epsilon.$$

As before writing

$$F_\sigma(t,u) = \sum_{j=0}^{m} \{\frac{n^{j/2}}{j!}\left(\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})\right) \Delta^j f_\sigma(u)\},$$

$$J_3(t) = \int_A^B W(n,t,u)(F_\sigma(t,u)-f_\sigma(t))\,du$$

$$- \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_0+2}}\,|u-t|^{m_0+2}\,F_\sigma(t,u)\,du$$

$$(5.1.19) \qquad = J_{31}(t) - J_{32}(t), \text{ say.}$$

A typical component of $J_{31}(t)$ is of the type

$$c\, n^{(j-r)/2} \int_A^B W(n,t,u)(t-u)^{j-r}\,(\Delta^j(f_\sigma(u)-f_\sigma(t)))\,du$$

(because $\Delta^j f_\sigma(t) = 0$ as Δ acts on u-part only)

$$= c\, n^{(j-r)/2} \Big\{ \sum_{s=0}^{j} \binom{j}{s} (-1)^{j-s} \Big\{ \int_A^B W(n,t,u)(t-u)^{j-r} \times$$

$$\times \big(f_\sigma(u+\frac{s}{n^{1/2}}) - f_\sigma(t)\big)\, du \Big\}$$

(5.1.20) $$= c\, n^{(j-r)/2} \Big\{ \sum_{s=0}^{j} \binom{j}{s} (-1)^{j-s} \Sigma_s(t) \Big\}, \text{ say},$$

where $0 \le r \le j-1$ and $c = c(j,r)$ is a scalar.

We estimate Σ_s as in the proof of Theorem 3.1.4. Applying Lemma 1.8.2 we obtain for a given $\epsilon > 0$

$$|\Sigma_s(t)| \le M_2 \frac{1}{n^{(j-r)/2}} (\epsilon + \frac{1}{n}), \quad t \in I_1.$$

Thus

(5.1.21) $$\|J_{31}(t)\|_{L_p(I_2)} \le M_2' (\epsilon + \frac{1}{n}).$$

A typical component of $J_{32}(t)$ is of the type

$$c_1\, n^{(j-r)/2} \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} (u-t)^{j-r} |u-t|^{m_o+2} \Delta^j f_\sigma(u)\, du$$

$$= T(t), \text{ say}.$$

Since f_σ is bounded over (A,B), we have by Corollary 1.8.2

$$|T(t)| \le \frac{M_3}{n^{(m_o+2)/2}} \|f_\sigma\|_{C[A,B]}, \quad t \in I_2.$$

Hence the condition $m \le m_o$ implies that

(5.1.22) $$\|J_{32}(t)\|_{L_p(I_2)} \le \frac{M_3'}{n^{(m+2)/2}} \|f_\sigma\|_{C[A,B]}.$$

Since $\epsilon > 0$ is arbitrary, the theorem follows from (5.1.16) to (5.1.22).

Corollary 5.1.6. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then

$$(5.1.23) \qquad ||f(t)-S_{n,m}(f,t)||_{L_p[A,B]} = o(1), \quad (n \to \infty).$$

Proceeding as in the proof of Corollary 4.1.5, (5.1.23) follows from Corollary 5.1.4 and Theorem 5.1.5.

5.2 ERROR ESTIMATES AND A DIRECT THEOREM

As in the case of the interpolatory modifications $P_{n,m}(\cdot,t)$ of Bernstein-Kantorovitch polynomials we show here that $\{S_{n,m}(\cdot,t)\}$ converges more rapidly for smoother functions in L_p-norm $(1 \leq p < \infty)$. We estimate rate of convergence in L_p-norm in terms of norms of derivatives of the function and also in terms of an (m+1)th integral modulus of smoothness of the function. Last result of this section is a Voronovskaja type asymptotic formula for the operators $S_{n,m}(\cdot,t)$.

Theorem 5.2.1. Let $1 < p < \infty$ and $f \in L_p[A,B]$. If f has $m+1$ derivatives over I_1 with $f^{(m)} \in A.C.(I_1)$ and $f^{(m+1)} \in L_p(I_1)$, then for sufficiently large values of n

$$(5.2.1) \qquad ||S_{n,m}(f,t)-f(t)||_{L_p(I_2)}$$

$$\leq M\left\{\frac{1}{n^{(m+1)/2}}||f^{(m+1)}||_{L_p(I_1)} + \frac{1}{n^{(m+2)/2}}||f||_{L_p[A,B]}\right\},$$

where M is a constant.

Proof. With the assumed hypothesis on f, for all $t \in I_2$ and $u \in I_1$ we can write

$$(5.2.2)\quad f(u) = \sum_{i=0}^{m} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{m!}\int_t^u (u-w)^m f^{(m+1)}(w)\,dw.$$

Let $\chi(u)$ be the characteristic function of $[a^*,b^*]$ where $a_1 < a^* < a_2 < b_2 < b^* < b_1$. Then for $t \in I_2$

$$S_{n,m}(f,t)-f(t) = \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} F(t,u)\,du - f(t)$$

$$= \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}}(F(t,u)-f(t))du - f(t)\int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}}|u-t|^{m_o+2}\,du$$

$$= \int_A^B \frac{(1-\chi(u)W(n,t,u)}{1+|u-t|^{m_o+2}} (F(t,u)-f(t))\,du$$

$$+ \int_A^B \frac{\chi(u)\ W(n,t,u)}{1+|u-t|^{m_o+2}}(F(t,u)-f(t))du - f(t)\int_A^B - \frac{W(n,t,u)}{1+|u-t|^{m_o+2}}|u-t|^{m_o+2}du$$

$$(5.2.3)\qquad = J_1(t) + J_2(t) + J_3(t), \text{ say.}$$

We first obtain a L_p-estimate of $J_2(t)$. It follows from from (5.2.2) and (3.1.3) that for $t \in I_2$

$$\chi(u)(F(t,u)-f(t)) = \frac{1}{m!}\chi(u)\left\{\sum_{j=0}^{m} \frac{n^{j/2}}{j!}\left(\prod_{i=0}^{j-1}\left(t-u-\frac{i}{n^{1/2}}\right)\right)\times\right.$$

$$\left.\times\left\{\Delta^j\left(\int_t^u (u-w)^m f^{(m+1)}(w)\,dw\right)\right\}\right\}.$$

This implies that after writing $\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})$ in the summation form and using identity

$$\Delta^j h(u) = \sum_{s=0}^{j} \binom{j}{s}(-1)^{j-s} h(u+\frac{s}{n^{1/2}}),$$

a typical component of $J_2(t)$ can be written as

$$c\, n^{(j-r-k)/2} \int_A^B \frac{X(u)\, W(n,t,u)}{1+|u-t|^{m_o+2}} (u-t)^{j-r} \times$$

$$\times \{ \int_t^{u+\frac{s}{n^{1/2}}} (u-w)^{m-k} f^{(m+1)}(w)dw \}\, du$$

$$= T_2(t), \text{ say,}$$

where $T_2(t) = T_2(t;j,r,s,k)$, $0 \leq j \leq m$, $0 \leq r \leq j-1$, $0 \leq s \leq j$, $0 \leq k \leq m$ and c is a scalar.

We rewrite $T_2(t)$ as

$$(5.2.4)\quad T_2(t) = c\, n^{(j-r-k)/2} \int_A^B \frac{X(u)W(n,t,u)}{1+|u-t|^{m_o+2}} (u-t)^{j-r} \times$$

$$\times \{ \int_t^u (u-w)^{m-k} f^{(m+1)}(w)dw + \int_u^{u+\frac{s}{n^{1/2}}} (u-w)^{m-k} f^{(m+1)}(w)dw \}\, du$$

$$= T_{21}(t) + T_{22}(t), \text{ say.}$$

Proposition 4.2.2 implies that

$$(5.2.5)\quad ||T_{21}||_{L_p(I_2)} \leq \frac{M_1}{n^{(m+1)/2}} ||f^{(m+1)}||_{L_p[a^*,b^*]}.$$

Also, proceeding as in the estimate of $T_{22}(t)$ in Theorem 3.2.1 we obtain a bound for $T_{22}(t)$ in this case; the only difference being that instead of applying the estimate

$$\int_0^1 K(n,t,u)|t-u|^{(j-r)p}\,dt \le \frac{M_2}{n^{(j-r)p/2}}$$

we use

$$\int_A^B W(n,t,u)|u-t|^{(j-r)p}\,dt \le \frac{M_2}{n^{(j-r)p/2}}$$

which follows from Corollary 1.8.10. Thus for large values of n

$$(5.2.6)\qquad ||T_{22}||_{L_p(I_2)} \le \frac{M_2'}{n^{(m+1)/2}}\,||f^{(m+1)}||_{L_p(I_1)}.$$

It follows from (5.2.4), (5.2.5) and (5.2.6) that

$$||T_2||_{L_p(I_2)} \le \frac{M_o}{n^{(m+1)/2}}\,||f^{(m+1)}||_{L_p(I_1)},$$

and hence

$$(5.2.7)\qquad ||J_2||_{L_p(I_2)} \le \frac{M_3}{n^{(m+1)/2}}\,||f^{(m+1)}||_{L_p(I_1)}.$$

Applying Corollary 1.8.2

$$J_3(t) \le \frac{M_3'}{n^{(m_o+2)/2}}\,|f(t)| \le \frac{M_3'}{n^{(m+2)/2}}\,|f(t)|,$$

for all $t \in I_2$. Hence

$$(5.2.8)\qquad ||J_3(t)||_{L_p(I_2)} \le \frac{M_4}{n^{(m+2)/2}}\,||f||_{L_p(I_2)}.$$

Now

$$J_1(t) = \int_A^B \frac{(1-\chi(u))W(n,t,u)}{1+|u-t|^{m_o+2}} F(t,u)\,du$$

$$- f(t) \int_A^B \frac{(1-\chi(u))W(n,t,u)}{1+|u-t|^{m_o+2}}\,du$$

$$= J_{11}(t) - J_{12}(t), \text{ say.}$$

Let $\delta = \min(a_2-a^*, b^*-b_2)$ and ℓ be a fixed positive number. Then presence of the factor $(1-\chi(u))$, by Corollary 1.8.2, implies that

$$|J_{12}(t)| \leq \frac{|f(t)|}{\delta^{2\ell}} \int_A^B (1-\chi(u))|u-t|^{2\ell}\, W(n,t,u)\,du$$

$$\leq \frac{M_4'}{n^\ell} |f(t)|, \quad \text{and hence}$$

$$||J_{12}||_{L_p(I_2)} \leq \frac{M_5}{n^\ell} ||f||_{L_p(I_2)}.$$

To obtain a bound for $||J_{11}||_{L_p(I_2)}$ we proceed as in the proof of estimate of J_2 in Theorem 5.1.3. We get

$$||J_{11}||_{L_p(I_2)} \leq \frac{M_5'}{n^\ell} ||f||_{L_p[A,B]}.$$

The L_p-bounds for $J_{11}(t)$ and $J_{12}(t)$ give the corresponding L_p-bound for $J_1(t)$ as

$$(5.2.9) \qquad ||J_1||_{L_p(I_2)} \leq \frac{M_6}{n^\ell} ||f||_{L_p[A,B]}.$$

The theorem now follows from (5.2.3), (5.2.7), (5.2.8) and (5.2.9) upon taking $\ell = (m+2)/2$ in (5.2.9).

Corollary 5.2.2. Let $A, B \in R$, $1 < p < \infty$ and $f \in L_p[A,B]$. If f has m+1 derivatives over $[A,B]$ with $f^{(m)} \in A.C.\ [A,B]$ and $f^{(m+1)} \in L_p[A,B]$, then for some constant M

$$(5.2.10)\quad ||S_{n,m}(f,t)-f(t)||_{L_p[A,B]}$$

$$\leq M\ \{\frac{1}{n^{(m+1)/2}}\ ||f^{(m+1)}||_{L_p[A,B]} + \frac{1}{n^{(m+2)/2}}\ ||f||_{L_p[A,B]}\}.$$

Proof. Proceeding as in the proof of above theorem and using second assertion (4.2.3) of Proposition 4.2.2 to obtain L_p-bound for the function $T_{21}(t)$ we can complete the proof.

Theorem 5.2.3. Let $f \in L_1[A,B]$. If f has m derivatives over I_1 with $f^{(m-1)} \in A.C.(I_1)$ and $f^{(m)} \in B.V.(I_1)$, then for sufficiently large values of n

$$(5.2.11)\quad ||S_{n,m}(f,t)-f(t)||_{L_1(I_2)}$$

$$\leq M\ \{\frac{1}{n^{(m+1)/2}}||f^{(m)}||_{B.V.(I_1)} + \frac{1}{n^{(m+2)/2}}||f||_{L_1[A,B]}\},$$

where M is a constant.

Proof. As in the proof of Theorem 5.2.1 we write

$$(5.2.12)\quad S_{n,m}(f,t)-f(t) = J_1(t) + J_2(t) + J_3(t),$$

where $J_1(t)$, $J_2(t)$ and $J_3(t)$ are given by (5.2.3).

Proceeding as in the case of L_p-estimates ($p > 1$) of $J_1(t)$ and $J_3(t)$ we obtain

(5.2.13) $\|J_1\|_{L_1(I_2)}$ and $\|J_3\|_{L_1(I_2)} \leq \frac{M_1}{n^{(m+2)/2}}\|f\|_{L_1[A,B]}$.

Now $f^{(m)} \in$ B.V.(I_1) implies by Theorem 14.1 of [61] that for all $u \in I_1$ and for almost all $t \in I_2$

(5.2.14) $$f(u) = \sum_{i=0}^{m} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{m!}\int_t^u (u-w)^m \, df^{(m)}(w).$$

As in the proof of (i) of Lemma 5.1.1, this implies that

$\chi(u)(F(t,u)-f(t))$

$$= \frac{1}{m!}\chi(u)\{\sum_{j=0}^{m} \frac{n^{j/2}}{j!}(\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}}))(\Delta^j(\int_t^u (u-w)^m df^{(m)}(w)))\}.$$

Putting this in the expression of $J_2(t)$ in (5.2.3) we find that a typical term is of the type

$$c\, n^{(j-r-k)/2} \int_A^B \frac{\chi(u)W(n,t,u)}{1+|u-t|^{m_o+2}} (t-u)^{j-r}\{\int_t^{u+\frac{s}{n^{1/2}}} (u-w)^{m-k} df^{(m)}(w)\} du$$

$$= T_3(t), \text{ say,}$$

where $0 \leq j \leq m$, $0 \leq r \leq j-1$, $0 \leq s \leq j$, $0 \leq k \leq m$ and c is a scalar.

This is further decomposed into two parts

$$T_3(t) = c\, n^{(j-r-k)/2} \int_A^B \frac{\chi(u)\, W(n,t,u)}{1+|u-t|^{m_o+2}} (t-u)^{j-r} \times$$

$$\times \{\int_t^u (u-w)^{m-k} df^{(m)}(w) + \int_u^{u+\frac{s}{n^{1/2}}} (u-w)^{m-k} df^{(m)}(w)\} du$$

(5.2.15) $= T_{31}(t) + T_{32}(t)$, say.

By Proposition 4.2.5

$$(5.2.16)\quad \|T_{31}(t)\|_{L_1(I_2)} \leq \frac{M_1'}{n^{(m+1)/2}} \|f^{(m)}\|_{B.V.[a^*,b^*]}$$

To obtain a L_p-bound for the function $T_{32}(t)$ we proceed as in the proof of estimate of $T_{22}(t)$ of Theorem 3.2.1. Thus using adjoint moment estimates of $S_n(.,t)$ given by Corollary 1.8.10 instead of adjoint moment estimates of $P_n(.,t)$, for large values of n we obtain

$$(5.2.17)\quad \|T_{32}(t)\|_{L_1(I_2)} \leq \frac{M_2}{n^{(m+1)/2}} \|f^{(m)}\|_{B.V.(I_1)}.$$

Thus, (5.2.15), (5.2.16) and (5.2.17) imply that

$$\|T_3(t)\|_{L_1(I_2)} \leq \frac{M_2'}{n^{(m+1)/2}} \|f^{(m)}\|_{B.V.(I_1)}$$

and hence

$$(5.2.18)\quad \|J_1(t)\|_{L_1(I_2)} \leq \frac{M_3}{n^{(m+1)/2}} \|f^{(m)}\|_{B.V.(I_1)}.$$

The theorem follows from (5.2.12), (5.2.13) and (5.2.18).

Corollary 5.2.4. Let $A,B \in \mathbb{R}$ and $f \in L_1[A,B]$. If f has m derivatives over the set $[A,B]$ with $f^{(m-1)} \in A.C.[A,B]$ and $f^{(m)} \in B.V.[A,B]$ then for some constant M

$$(5.2.19)\quad \|S_{n,m}(f,t)-f(t)\|_{L_1[A,B]}$$

$$\leq M\left\{\frac{1}{n^{(m+1)/2}} \|f^{(m)}\|_{B.V.[A,B]}\right.$$

$$\left. + \frac{1}{n^{(m+2)/2}} \|f\|_{L_1[A,B]}\right\},$$

Proof. Proceeding as in the proof of the above theorem and making use of (4.2.13) of Proposition 4.2.5 to obtain an L_p-bound for $T_{31}(t)$ we obtain (5.2.19).

Theorem 5.2.5. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then, for all sufficiently large values of n,

$$(5.2.20) \quad ||S_{n,m}(f,t)-f(t)||_{L_p(I_2)}$$
$$\leq M \{\omega_{m+1}(f,n^{-1/2},p,I_1) + \frac{1}{n^{(m+2)/2}} ||f||_{L_p[A,B]}\},$$

where M is a certain constant independent of n and f.

Proof. with a^*,b^* as before and $f_{n,m+1}$ being the Steklov mean of (m+1)th order corresponding to f(u), we have

$$||S_{n,m}(f,t)-f(t)||_{L_p(I_2)} \leq ||S_{n,m}(f-f_{n,m+1},t)||_{L_p(I_2)}$$
$$+ ||S_{n,m}(f_{n,m+1},t)-f_{n,m+1}(t)||_{L_p(I_2)} + ||f_{n,m+1}(t)-f(t)||_{L_p(I_2)}.$$

Applying Theorems 5.1.3, 5.2.1 ($p > 1$) and 5.2.3 ($p=1$) to the first and second terms on the right hand side of the above inequality and upon taking $\ell = (m+2)/2$, for large values of n we obtain

$$||S_{n,m}(f,t)-f(t)||_{L_p(I_2)}$$
$$\leq M_1 \{||f-f_{n,m+1}||_{L_p[a^*,b^*]} + \frac{1}{n^{(m+1)/2}}||f^{(m+1)}_{n,m+1}||_{L_p[a^*,b^*]} +$$

$$+ \frac{1}{n^{(m+2)/2}} \quad \|f-f_{\eta,m+1}\|_{L_p[A,B]} + \frac{1}{n^{(m+2)/2}}\|f_{\eta,m+1}\|_{L_p[A,B]} \}.$$

Using the estimates (1.3.2), (1.3.3) and (1.3.4) of Lemma 1.3.1 and taking $\eta = n^{-1/2}$ we obtain the result.

Theorem 5.2.6. Let $f \in C_0^{m+1}$ with supp $f \subset I_1$. Then

$$(5.2.20) \quad S_{n,m}(f,t)-f(t) = \frac{(-1)^m q_{m+1}(t)}{(m+1)!\ n^{(m+1)/2}} f^{(m+1)}(t) + o(\frac{1}{n^{(m+1)/2}}), \quad (n \to \infty)$$

and

$$(5.2.21) \quad S_{n,m+1}(f,t)-f(t) = o(\frac{1}{n^{(m+1)/2}}), \quad (n \to \infty),$$

uniformly in $t \in I_1$, where $q_{m+1}(t)$ is as defined in Corollary 5.1.2.

Proof. Given that f is m+1 times continuously differentiable, we have for some ξ lying between u and t

$$f(u) = \sum_{i=0}^{m+1} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{(u-t)^{m+1}}{(m+1)!} (f^{(m+1)}(\xi)-f^{(m+1)}(t)).$$

Applying the operator $S_{n,m}(.,t)$ on both sides of the above identity and using (i) of Lemma 5.1.1, we obtain

$$S_{n,m}(f,t)-f(t) = f(t)(S_{n,m}(1,t)-1) + \frac{f^{(m+1)}(t)}{(m+1)!} S_{n,m}((u-t)^{m+1},t)$$

$$+ \frac{1}{(m+1)!} S_{n,m}((u-t)^{m+1}(f^{(m+1)}(\xi)-f^{(m+1)}(t)),t)$$

$$(5.2.22) \qquad = J_1(t) + J_2(t) + J_3(t), \text{ say.}$$

By Corollary 1.8.2 for $t \in I_1$

$$|J_1(t)| = |f(t)\{\int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} du\}|$$

$$\leq \| f \|_{C(I_1)} \{\int_A^B W(n,t,u) |u-t|^{m_o+2} du\}$$

$$(5.2.23) \qquad \leq \frac{M_1}{n^{(m_o+2)/2}} \leq \frac{M_1}{n^{(m+2)/2}} .$$

Applying Corollary 5.1.2 to $J_2(t)$, we obtain

$$(5.2.24) \quad J_2(t) = \frac{(-1)^m q_{m+1}(t)}{(m+1)!\; n^{(m+1)/2}} f^{(m+1)}(t) + o(\frac{1}{n^{(m+1)/2}}), (n \to \infty).$$

It remains to estimate $J_3(t)$, a typical component of which is of the type

$$c\, n^{(j-r-k)/2} \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} (u-t)^{m+1+j-r-k}(f^{(m+1)}(\xi_s)-f(t))\, du$$

$$(5.2.25) \qquad = T(t), \text{ say,}$$

where $0 \leq j \leq m$, $0 \leq r \leq j-1$, $0 \leq k \leq m+1$, $0 \leq s \leq j$, ξ_s lies between $u + \frac{s}{n^{1/2}}$ and t, and c is a scalar.

Let $\epsilon > 0$ be given. We proceed as in the proof of Theorem 3.2.6 to show that

$$(5.2.26) \qquad |T(t)| \leq \frac{M_2}{n^{(m+1)/2}} (\epsilon + \frac{1}{n}).$$

Since $\epsilon > 0$ is arbitrary, we find from (5.2.25) and (5.2.26) that

$$(5.2.27) \qquad J_3(t) = o\left(\frac{1}{n^{(m+1)/2}}\right), \quad (n \to \infty),$$

uniformly in $t \in I_1$.

The first assertion of the theorem now follows from (5.2.22), (5.2.23), (5.2.24) and (5.2.27).

Proceeding as in the proof of (5.2.20) and using (i) of Lemma 5.1.1

$$S_{n,m+1}((u-t)^{m+1},t) = 0,$$

we obtain (5.2.21).

5.3 INVERSE THEOREM

In this section an inverse theorem for $\{S_{n,m}(\cdot,t)\}$ in L_p-norm $(1 \leq p < \infty)$ is proved.

Theorem 5.3.1. Let $0 < \alpha < m+1$, $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then

$$(5.3.1) \qquad \|S_{n,m}(f,t)-f(t)\|_{L_p(I_1)} = O(n^{-\alpha/2}), \quad (n \to \infty)$$

implies that

$$(5.3.2) \qquad \omega_{m+1}(f,\tau,p,I_2) = O(\tau^{\alpha}), \quad (\tau \to 0).$$

The proof of the theorem makes use of the following two lemmas which we prove first.

Lemma 5.3.2. Let $1 \leq p < \infty$ and $h \in L_p[A,B]$. Then for any fixed positive number ℓ and sufficiently large value of n

$$(5.3.3) \quad ||S_{n,m}(|u-t|\cdot|h(u)|,t)||_{L_p(I_2)}$$

$$\leq M\{n^{-1/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p[A,B]}\},$$

where M is a constant independent of n and h.

<u>Proof</u>. A typical component of $S_{n,m}(|u-t|\cdot|h(u)|,t)$ is of the type

$$c\, n^{(j+k-r-1)/2} \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} (t-u)^{j-r}|u-t|^k\, h(u+\frac{s}{n^{1/2}})\, du$$

$$= T(t), \text{ say,}$$

where $T(t) = T(t;j,r,s,k)$,

$0 \leq j \leq m$, $0 \leq r \leq j-1$ (when $j = 0$, $r = 0$), $0 \leq s \leq j$, $k = 0$ or 1, and c is a scalar.

Let $\chi(u)$ be characteristic function of $[a^*,b^*]$ where $a_1 < a^* < a_2 < b_2 < b^* < b_1$. Also let $\theta = j+k-r-1$.

By Jensen's inequality

$$\int_{a_2}^{b_2} |T(t)|^p\, dt$$

$$\leq (|c|n^{\theta/2})^p \int_{a_2}^{b_2}\int_A^B \frac{W(n,t,u)}{(1+|u-t|^{m_o+2})^p}|u-t|^{(\theta+1)p}|h(u+\frac{s}{n^{1/2}})|^p\, du\, dt$$

$$= (|c|n^{\theta/2})^p\{\int_{a_2}^{b_2}\int_A W(n,t,u)\{\frac{\chi(u)\,|u-t|^{(\theta+1)p}}{(1+|u-t|^{m_o+2})^p}|h(u+\frac{s}{n^{1/2}})|^p +$$

$$+ \frac{(1-\chi(u))|u-t|^{(\theta+1)p}}{(1+|u-t|^{m_o+2})^p} |h(u + \frac{s}{n^{1/2}})|^p \} du\, dt \}$$

(5.3.4) $\quad = J_1 + J_2$, say.

It follows from the proof of estimates of J_1 and J_2 of Theorem 5.1.3 that for any fixed positive number ℓ and for sufficiently large values of n there holds

$$J_1 \leq M_1\, n^{-p/2}\, ||h||^p_{L_p(I_1)}$$

and
$$J_2 \leq M_1\, n^{-\ell p}\, ||h||^p_{L_p[A,B]} .$$

The estimates of J_1 and J_2 imply by (5.3.4) that

$$||T||_{L_p(I_2)} \leq M_1' \{ n^{-1/2}\, ||h||_{L_p(I_1)} + n^{-\ell}\, ||h||_{L_p[A,B]} \} .$$

Hence, $T(t)$ being a typical component, we obtain (5.3.3).

For $h \in L_p[A,B]$ where $1 \leq p < \infty$ and supp $h \subset (A,B)$, we define

(5.3.5) $$\bar{S}_{n,m}(h,t) = \int_A^B W(n,t,u) \times$$

$$\times \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}}))\, \Delta^j\, h(u) \}\, du .$$

<u>Lemma 5.3.3.</u> Let $1 \leq p < \infty$, $h \in L_p[A,B]$ and supp $h \subset [a,b]$ where $A < a < b < B$. Then

(5.3.6) $$||\bar{S}^{(m+1)}_{n,m}(h,t)||_{L_p[a,b]} \leq M\, n^{(m+1)/2} ||h||_{L_p[a,b]} ,$$

where M is a constant independent of n and h.

Moreover, if h has m+1 derivatives over $[a,b]$ with $h^{(m)} \in A.C.[a,b]$ and $h^{(m+1)} \in L_p[a,b]$ then

$$(5.3.7) \quad \|\bar{S}_{n,m}^{(m+1)}(h,t)\|_{L_p[a,b]} \le M' \|h^{(m+1)}\|_{L_p[a,b]},$$

the constant M' is independent of n and h.

Proof. As in the case of $P_{n,m}^{(m+1)}(h,t)$ in Lemma 3.3.3, a typical component of $\bar{S}_{n,m}^{(m+1)}(h,t)$ is of the type

$$c\, n^{r_1/2} \{ \sum_{i_1, j_1} n^{i_1+j_1} \{ \frac{a_{i_1 j_1}^{(m+1-k)}(t)}{(p(t))^{m+1-k}} \times$$

$$\times \{ \int_A^B W(n,t,u)(u-t)^{j_1+r_2} \Delta^{r_3} h(u)du \}\}\}$$

$$= T_4(t), \text{ say,}$$

where $0 \le k \le m$, $2i_1+j_1 \le m+1-k$, $k \le r_3 \le m$, $k \le r_1 \le r_3$, $0 \le r_2 \le r_3-k$, $r_1-r_2 = k$, c is a scalar and $a_{ij}^{(m)}(t)$ are the polynomials occuring in Lemma 1.8.3.

Since p(t) is bounded over I_2, by Lemma 4.3.2 we obtain

$$\|T_4(t)\|_{L_p[a,b]} \le M_1\, n^{(m+1)/2} \|h\|_{L_p[a,b]}.$$

This proves (5.3.6).

Next, writing

$$h(u) = \sum_{i=0}^{m} \frac{(u-t)^i}{i!} h^{(i)}(t) + \frac{1}{m!} \int_t^u (u-w)^m h^{(m+1)}(w)\, dw,$$

and operating by $\bar{S}_{n,m}^{(m+1)}(\cdot,t)$ on both sides we get

$$\bar{S}_{n,m}^{(m+1)}(h,t) = \frac{1}{m!}\,\bar{S}_{n,m}^{(m+1)}\Big(\int_t^u (u-w)^m\, h^{(m+1)}(w)dw,\ t\Big),$$

since $\bar{S}_{n,m}^{(m+1)}((u-t)^i,t) = 0, \quad i = 0,1,\ldots,m.$

As in Lemmas 3.3.3 and 4.3.3 a typical component of $\bar{S}_{n,m}^{(m+1)}(h,t)$ can be represented by

$$c\; n^{(r_1+r_4-m)/2}\Big\{\sum_{i_1,j_1} n^{i_1+j_1}\Big\{\frac{a_{i_1j_1}^{(m+1-k)}(t)}{(p(t))^{m+1-k}}\Big\{\int_A^B W(n,t,u)\,\times$$

$$\times(u-t)^{j_1+r_2}\Big\{\int_t^{u+\frac{r_3}{n^{1/2}}}(u-w)^{r_4}\,h^{(m+1)}(w)dw\Big\}du\Big\}\Big\}\Big\}$$

$$= T_5(t),\text{say},$$

where i_1, j_1, r_1, r_2 are as in $T_4(t)$ and $0 \leq r_3, r_4 \leq m$.

Proceeding as in the proofs of estimates of $T_2(t)$ and $T_3(t)$ in Theorems 5.2.1 and 5.2.3 respectively, for the cases $1 < p < \infty$ and $p = 1$, we obtain

$$||T_5(t)||_{L_p[a,b]} \leq M_2\, ||h^{(m+1)}||_{L_p[a,b]}.$$

This implies that

$$||\bar{S}_{n,m}^{(m+1)}(h,t)||_{L_p[a,b]} \leq M'\, ||h^{(m+1)}||_{L_p[a,b]},$$

completing proof of (5.3.7).

Proof of Theorem 5.3.1. We choose points $x_i, y_i; i=1,2,3,4$ such that $a_1 < x_i < a_2$, $b_2 < y_i < b_1$, $x_i < x_{i+1}$ and $y_{i+1} < y_i$. We choose a function $g \in C_o^{m+1}$ with supp $g \subset (x_3, y_3)$ and $g(t) = 1$ for $t \in [x_4, y_4]$.

By (5.3.5), if $\bar{F}(t,u) = \sum_{j=0}^{m} \{\frac{n^{j/2}}{j!} \left(\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}})\right) \Delta^j \bar{f}(u)\}$,

$$S_{n,m}(\bar{f},t) = \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} \bar{F}(t,u)\,du$$

$$= \int_A^B W(n,t,u)\,\bar{F}(t,u)du$$

$$- \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} \bar{F}(t,u)\,du$$

$$(5.3.8) \qquad = \bar{S}_{n,m}(\bar{f},t) - \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} \bar{F}(t,u)\,du$$

By (5.3.8), for sufficiently small positive values of γ

$$||\Delta_\gamma^{m+1} \bar{f}(t)||_{L_p[x_3,y_3]}$$

$$\leq ||\Delta_\gamma^{m+1}\{\bar{f}(t)-S_{n,m}(\bar{f},t)\}||_{L_p[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{m+1} S_{n,m}(\bar{f},t)||_{L_p[x_3,y_3]}$$

$$\leq ||\Delta_\gamma^{m+1}\{\bar{f}(t)-S_{n,m}(\bar{f},t)\}||_{L_p[x_3,y_3]}$$

$$+ \|\Delta_\gamma^{m+1} \bar{S}_{n,m}(\bar{f},t)\|_{L_p[x_3,y_3]}$$

$$+ \|\Delta_\gamma^{m+1} \{\int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} \bar{F}(t,u)du\}\|_{L_p[x_3,y_3]}$$

$$(5.3.9) \qquad = J_1 + J_2 + J_3, \text{ say.}$$

Proceeding as in the proof of Theorem 4.3.1

$$J_2 \le \gamma^{m+1} \|\bar{S}_{n,m}^{(m+1)}(\bar{f},t)\|_{L_p[x_3,y_3+(m+1)\gamma]}$$

$$\le M_1 \gamma^{m+1} \{n^{(m+1)/2}\|\bar{f}-\bar{f}_{\eta,m+1}\|_{L_p[x_3,y_3]}$$

$$+ \|f_{\eta,m+1}^{(m+1)}\|_{L_p[x_3,y_3]}\},$$

by Lemma 5.3.3 where $\eta > 0$ is sufficiently small.

This, in conjunction with the estimates (1.3.2) and (1.3.3), implies that

$$(5.3.10) \quad J_2 \le M_1' \gamma^{m+1}(n^{(m+1)/2} + \eta^{-(m+1)})\omega_{m+1}(\bar{f},\eta,p,[x_2,y_2]).$$

We obtain a bound for J_3 as follows.

Let

$$T(t) = c\, n^{(j-r)/2} \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}}|u-t|^{m_o+2}(t-u)^{j-r}(\Delta^j \bar{f}(u))du,$$

where $0 \le r \le j-1$ and c is a scalar.

It follows from Lemma 4.3.2 that

$$||T(t)||_{L_p[x_3,y_3+(m+1)\gamma]} \leq \frac{M_2}{n^{(m_0+2)/2}} ||\bar{f}||_{L_p[x_3,y_3]}$$

$$\leq \frac{M_2}{n^{(m+2)/2}} ||\bar{f}||_{L_p[x_3,y_3]}.$$

Consequently

$$(5.3.11) \qquad J_3 \leq \frac{M_2'}{n^{(m+2)/2}} ||\bar{f}||_{L_p[x_3,y_3]}.$$

Thus, by (5.3.9), (5.3.10) and (5.3.11), we obtain

$$(5.3.12) \qquad ||\Delta_\gamma^{m+1} \bar{f}(t)||_{L_p[x_3,y_3]}$$

$$\leq ||\Delta_\gamma^{m+1} \{\bar{f}(t)-S_{n,m}(\bar{f},t)\}||_{L_p[x_3,y_3]}$$

$$+ M_3 \; \gamma^{m+1}(n^{(m+1)/2}+n^{-(m+1)})\omega_{m+1}(\bar{f},n,p,[x_2,y_2])$$

$$+ M_3 \; n^{-(m+2)/2} ||\bar{f}||_{L_p[x_3,y_3]}.$$

Now to complete the proof of the theorem, as in the case of the earlier inverse theorems in Chapters II, III and IV, we are left to prove that

$$(5.3.13) \quad ||\Delta_\gamma^{m+1} \{\bar{f}(t)-S_{n,m}(\bar{f},t)\}||_{L_p[x_3,y_3]} = O(n^{-\alpha/2}), (n \to \infty).$$

We first prove (5.3.13) when $\alpha \leq 1$.

Using (5.3.1)

$$||S_{n,m}(fg,t)-(fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq ||g(t)\{S_{n,m}(f,t)-f(t)\}||_{L_p[x_3,y_3]}$$

$$+ ||S_{n,m}(f(u)(g(u)-g(t)),t)||_{L_p[x_3,y_3]}$$

$$(5.3.14) \quad \leq \frac{M_3'}{n^{\alpha/2}} + ||S_{n,m}(f(u)(u-t)g'(\xi),t)||_{L_p[x_3,y_3]}.$$

(where ξ lies between u and t).

Proceeding as in the proof of Lemma 5.3.2, for any fixed positive number ℓ and large values of n

$$||S_{n,m}(f(u)(u-t)g'(\xi),t)||_{L_p[x_3,y_3]}$$

$$(5.3.15) \quad \leq M_4 \{n^{-1/2}||f||_{L_p[x_2,y_2]} + n^{-\ell}\,||f||_{L_p[A,B]}\}.$$

Combining (5.3.14) and (5.3.15) we obtain

$$||S_{n,m}(fg,t)-(fg)(t)||_{L_p[x_3,y_3]} = O(n^{-\alpha/2}), \quad (n \to \infty),$$

proving (5.3.13).

Next, assuming that for some $r \leq m$, the theorem holds for those values of α satisfying $r-1 \leq \alpha < r$, we show that this is also true when $r \leq \alpha < r+1$.

Using (5.3.1)

$$(5.3.16)\quad \|S_{n,m}(fg,t)-(fg)(t)\|_{L_p[x_3,y_3]}$$
$$\leq \frac{M_3'}{n^{\alpha/2}} + \|S_{n,m}(f(u)(g(u)-g(t)),t)\|_{L_p[x_3,y_3]}.$$

Now,

$$\|S_{n,m}(f(u)(g(u)-g(t)),t)\|_{L_p[x_3,y_3]}$$
$$\leq \|S_{n,m}((f(u)-f_{\eta,m+1}(u))(g(u)-g(t)),t)\|_{L_p[x_3,y_3]}$$
$$+ \|S_{n,m}((f_{\eta,m+1}(u)-f_{\eta,m+1}(t))(g(u)-g(t)),t)\|_{L_p[x_3,y_3]}$$
$$+ \|S_{n,m}(f_{\eta,m+1}(t)(g(u)-g(t)),t)\|_{L_p[x_3,y_3]}$$

$$(5.3.17)\quad = J_1 + J_2 + J_3, \text{ say.}$$

As in (5.3.15), for any fixed positive number ℓ

$$J_1 \leq M_4 \{n^{-1/2}\|f-f_{\eta,m+1}\|_{L_p[x_2,y_2]} + n^{-\ell}\|f-f_{\eta,m+1}\|_{L_p[A,B]}\}.$$

Applying (1.3.3) and (1.3.4) to the right hand side of the above inequality and taking $\ell = (m+1)/2$

$$(5.3.18)\quad J_1 \leq M_4' \{n^{-1/2}\,\omega_{m+1}(f,\eta,p,[x_1,y_1])$$
$$+ n^{-(m+1)/2}\|f\|_{L_p[A,B]}\}.$$

By Theorem 5.2.6 and the estimate (1.3.4)

(5.3.19) $$J_3 \leq \frac{M_5}{n^{(m+1)/2}} ||f||_{L_p[A,B]}.$$

We can write for some ξ lying between u and t

$$(f_{n,m+1}(u)-f_{n,m+1}(t))(g(u)-g(t))$$

$$= \{\sum_{i=1}^{m} \frac{(u-t)^i}{i!} f^{(i)}_{n,m+1}(t) + \frac{1}{m!}\int_t^u (u-w)^m f^{(m+1)}_{n,m+1}(w)dw\} \times$$

$$\times \{\sum_{i=1}^{m-1} \frac{(u-t)^i}{i!} g^{(i)}(t) + \frac{(u-t)^m}{m!} g^{(m)}(\xi)\}$$

$$= \frac{1}{m!}\{\sum_{i=1}^{m} \frac{1}{i!} f^{(i)}_{n,m+1}(t) g^{(m)}(\xi)(u-t)^{m+i}\}$$

$$+ \{\sum_{i=1}^{m}\sum_{j=1}^{m-1} \{\frac{g^{(j)}(t)}{i!\,j!} f^{(i)}_{n,m+1}(t)(u-t)^{i+j}\}\}$$

$$+ \frac{1}{m!}\{\sum_{i=1}^{m-1}\{\frac{g^{(i)}(t)}{i!}(u-t)^i\{\int_t^u (u-w)^m f^{(m+1)}_{n,m+1}(w)dw\}\}\}$$

$$+ \frac{1}{(m!)^2}\{g^{(m)}(\xi)(u-t)^m\{\int_t^u (u-w)^m f^{(m+1)}_{n,m+1}(w)dw\}\}$$

$$= J_{2,1}(u,t) + J_{2,2}(u,t) + J_{2,3}(u,t) + J_{2,4}(u,t), \text{ say.}$$

Therefore,

(5.3.20) $$J_2 \leq \sum_{i=1}^{4} ||S_{n,m}(J_{2,i}(u,t),t)||_{L_p[x_3,y_3]}.$$

As before a typical component of $S_{n,m}(J_{2,1}(u,t),t)$ is of the type

$$c\, f^{(i)}_{n,m+1}(t)\, n^{(j-r-s)/2} \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} (u-t)^{i+j+m-r-s}\, g^{(m)}(\xi_j)\, du$$

$$= T(t), \text{ say,}$$

where $1 \leq i \leq m$, $0 \leq j \leq m$, $0 \leq r \leq j-1$, $0 \leq s \leq m+i$, ξ_j lies between $u + \frac{j}{n^{1/2}}$ and t, and c is a scalar.

Applying Corollary 1.8.2

$$|T(t)| \leq \frac{M_5'}{n^{(m+i)/2}} |f^{(i)}_{n,m+1}(t)|,$$

and hence,

$$||S_{n,m}(J_{2,1}(u,t),t)||_{L_p[x_3,y_3]}$$

$$(5.3.21) \qquad \leq M_6 \{ \sum_{i=1}^{m} \frac{1}{n^{(m+i)/2}} ||f^{(i)}_{n,m+1}||_{L_p[x_3,y_3]} \}.$$

From Lemmas 5.1.1 and 1.8.1,

$$(5.3.22) \quad ||S_{n,m}(J_{2,2}(u,t),t)||_{L_p[x_3,y_3]}$$

$$\leq M_6' \{ \sum_{i=1}^{m} \sum_{\substack{j=1 \\ i+j>m}}^{m-1} \frac{1}{n^{(i+j)/2}} ||f^{(i)}_{n,m+1}||_{L_p[x_3,y_3]} \}.$$

Next, we obtain L_p-bounds for $S_{n,m}(J_{2,3}(u,t),t)$ and $S_{n,m}(J_{2,4}(u,t),t)$. We note that a typical component of

$$S_{n,m}(g^{(m)}(\xi)(u-t)^k \{\int_t^u (u-w)^m f^{(m+1)}_{n,m+1}(w)dw\},t)$$

is of the type

$$c\, n^{(\theta-k)/2} \int_A^B \Big\{ \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} (u-t)^{\theta} g^{(m)}(\xi_{r_2}) \times$$

$$\times \Big(\int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m f_{n,m+1}^{(m+1)}(w)dw \Big) \Big\} du$$

(5.3.23) $= T(t)$, say,

where $\theta = j+r_3-r_1$, $0 \leq j \leq m$, $0 \leq r_1 \leq j-1$, $0 \leq r_2 \leq j$, $0 \leq r_3 \leq k$, ξ_{r_2} lies between $u + \frac{r_2}{n^{1/2}}$ and t and c is a scalar.

Let $\chi(u)$ be the characteristic function of $[x_2, y_2]$. Then, denoting the expression inside the curly brackets in (5.3.23) by $\Phi(t,u)$,

$$T(t) = c\, n^{(\theta-k)/2} \{ \int_A^B \chi(u)\Phi(t,u)du + \int_A^B (1-\chi(u))\Phi(t,u)du \},$$

$$= T_6(t) + T_7(t), \text{ say,}$$

It follows from the estimate of $T_2(t)$ in Theorem 5.2.1 that

$$||T_6(t)||_{L_p[x_3,y_3]} \leq \frac{M_7}{n^{(m+k+1)/2}} ||f_{n,m+1}^{(m+1)}||_{L_p[x_2,y_2]}.$$

It is easily seen that for any fixed positive number ℓ

$$||T_7(t)||_{L_p[x_3,y_3]} \leq \frac{M_7'}{n^{\ell}} ||f_{n,m+1}^{(m+1)}||_{L_p[A,B]}.$$

The L_p-bounds for $T_6(t)$ and $T_7(t)$ give the corresponding L_p-bound for the function $T(t)$, which, in turn, implies that

$$(5.3.24)\quad \|S_{n,m}\,(g^{(m)}(\xi)(u-t)^k \times$$

$$\times (\int_t^u (u-w)^m\, f_{n,m+1}^{(m+1)}(w)dw),t)\|_{L_p[x_3,y_3]}$$

$$\leq M_8\, \{\frac{1}{n^{(m+k+1)/2}}\, \|f_{n,m+1}^{(m+1)}\|_{L_p[x_2,y_2]}$$

$$+\frac{1}{n^{\ell}}\, \|f_{n,m+1}^{(m+1)}\|_{L_p[A,B]}\}.$$

Therefore, it follows from (5.3.24) that

$$(5.3.25)\quad \|S_{n,m}(J_{2,3}(u,t),t)\|_{L_p[x_3,y_3]}$$

$$\leq M_8'\, \{\frac{1}{n^{(m+2)/2}}\, \|f_{n,m+1}^{(m+1)}\|_{L_p[x_2,y_2]}$$

$$+\frac{1}{n^{\ell}}\, \|f_{n,m+1}^{(m+1)}\|_{L_p[A,B]}\}$$

and that

$$(5.3.26)\quad \|S_{n,m}(J_{2,4}(u,t),t)\|_{L_p[x_3,y_3]}$$

$$\leq M_8'\, \{\frac{1}{n^{(2m+1)/2}}\, \|f_{n,m+1}^{(m+1)}\|_{L_p[x_2,y_2]}$$

$$+\frac{1}{n^{\ell}}\, \|f_{n,m+1}^{(m+1)}\|_{L_p[A,B]}\}.$$

Combining inequalities (5.3.21), (5.3.22), (5.3.25) and (5.3.26) we see from (5.3.20) that

$$(5.3.27)\quad J_2 \le M_9 \{ \frac{1}{n^{(m+2)/2}} \|f^{(m+1)}_{n,m+1}\|_{L_p[x_2,y_2]}$$

$$+ \frac{1}{n^{(m+1)/2}} (\sum_{i=1}^{m} \|f^{(i)}_{n,m+1}\|_{L_p[x_3,y_3]}) + \frac{1}{n^{\ell}} \|f^{(m+1)}_{n,m+1}\|_{L_p[A,B]} \}.$$

Applying estimate (1.2.3) and after taking $\ell = m+1$ we obtain

$$(5.3.28)\quad J_2 \le M_9' \{ \frac{1}{n^{(m+2)/2}} \|f^{(m+1)}_{n,m+1}\|_{L_p[x_2,y_2]}$$

$$+ \frac{1}{n^{(m+1)/2}} \|f^{(m)}_{n,m+1}\|_{L_p[x_3,y_3]} + \frac{1}{n^{(m+1)/2}} \|f_{n,m+1}\|_{L_p[x_3,y_3]}$$

$$+ \frac{1}{n^{m+1}} \|f^{(m+1)}_{n,m+1}\|_{L_p[A,B]} \}$$

For sufficiently small $\eta > 0$ this by (1.3.2), (1.3.4) and (1.3.5), gives that

$$(5.3.29)\quad J_2 \le M_{10} \{ \frac{1}{\eta^{m+1}} \frac{1}{n^{(m+2)/2}} \omega_{m+1}(f,\eta,p,[x_1,y_1])$$

$$+ \frac{1}{\eta^{m}} \frac{1}{n^{(m+1)/2}} \omega_m(f,\eta,p,[x_1,y_1])$$

$$+ (\frac{1}{n^{(m+1)/2}} + \frac{1}{\eta^{m+1}} \frac{1}{n^{m+1}}) \|f\|_{L_p[A,B]} \}.$$

The induction hypothesis and Corollary 1.3.4 imply that

$$(5.3.30)\quad \omega_{m+1}(f,\eta,p,[x_1,y_1]) = O(\eta^{\alpha-1}), \quad (\eta \to 0),$$

and

$$(5.3.31)\quad \omega_m(f,\eta,p,[x_1,y_1]) = O(\eta^{\alpha-1}), \quad (\eta \to 0).$$

Finally it follows from (5.3.18), (5.3.19), (5.3.29), (5.3.30) and (5.3.31) after taking $\eta = n^{-1/2}$ that

(5.3.32) $$J_1,\ J_2 \text{ and } J_3 \leq \frac{M'_{10}}{n^{\alpha/2}}\ .$$

Combining inequalities (5.3.16), (5.3.17) and (5.3.32) we see that (5.3.13) holds good in this case also. This completes the proof of the theorem.

5.4 SATURATION THEOREM

In this section we consider the saturation behaviour of the sequence $\{S_{n,m}(\cdot,t)\}$ of operators. The main result shows that these operators are saturated with the same order $O(n^{-(m+1)/2})$ and the saturation and trivial classes, as has been the case with the operators $P_{n,m}(\cdot,t)$, the only difference being that for $P_{n,m}(\cdot,t)$ the interval is (0,1) while it is (A,B) for the present operators.

Theorem 5.4.1. Let $1 \leq p < \infty$ and $f \in L_p[A,B]$. Then, in the following statements, the implications "(i) ==> (ii) ==> (iii)" and "(iv) ==> (v) ==> (vi)" hold :

(i) $||S_{n,m}(f,t)-f(t)||_{L_p(I_1)} = O(n^{-(m+1)/2}),\ (n \to \infty);$

(ii) f coincides a.e. on I_2 with a function F having m+1 derivatives such that (a) when $p > 1$, $F^{(m)} \in A.C.(I_2)$ and $F^{(m+1)} \in L_p(I_2)$, (b) when $p = 1$, $F^{(m-1)} \in A.C.(I_2)$ and $F^{(m)} \in B.V.(I_2)$;

(iii) $\|S_{n,m}(f,t)-f(t)\|_{L_p(I_3)} = O(n^{-(m+1)/2}), \quad (n \to \infty);$

(iv) $\|S_{n,m}(f,t)-f(t)\|_{L_p(I_1)} = o(n^{-(m+1)/2}), \quad (n \to \infty);$

(v) f coincides a.e. on I_2 with a polynomial of degree m;

(vi) $\|S_{n,m}(f,t)-f(t)\|_{L_p(I_3)} = o(n^{-(m+1)/2}), \quad (n \to \infty).$

Note. The implication "(ii) ==> (iii)" holds in view of Theorems 5.2.1 and 5.2.3 for the cases $p > 1$ and $p = 1$ respectively. And "(v) ==> (vi)" follows from Theorem 5.2.6.

In the proofs of the saturation Theorems 2.4.1, 3.4.1 and 4.4.1 of the previous chapters the most important ingredients have been the inner product inequality Lemmas 2.4.2, 3.4.2 and 4.4.2, respectively. Once such a lemma has been established the proof of theorem in the linear combinations case follows the pattern of Theorem 2.4.1 while the same in the interpolatory case runs parallel to that of Theorem 3.4.1. For this reason, as has been done in the case of Theorem 4.4.1, here also we restrict ourselves to proving the following corresponding inner product inequality lemma only.

Lemma 5.4.2 : Let $1 \leq p < \infty$ and $h \in L_p[A,B]$ where h has compact support $\subset (A,B)$. Further, let h have m derivatives with (m-1)th derivative absolutely continuous and the mth derivative belonging to $L_p[A,B]$. Then, for each m+1 times continuously differentiable function g having a compact support inside (A,B)

$$(5.4.1)\quad |< S_{n,m}(h,t)-h(t),g(t) >|$$

$$\leq \frac{M}{n^{(m+1)/2}} \{ ||h^{(m)}||_{L_1[A,B]} + ||h||_{L_1[A,B]} \},$$

M being a constant independent of n and h.

Proof. We have with

$$F(t,u) = \sum_{j=0}^{m} \{ \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta^j h(u) \}$$

as before,

$$< S_{n,m}(h,t),g(t) > = \int_A^B S_{n,m}(h,t)g(t)\,dt$$

$$= \int_A^B \bar{S}_{n,m}(h,t)g(t)\,dt - \int_A^B\int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} \times$$

$$\times |u-t|^{m_o+2} F(t,u)g(t)\,du\,dt$$

$$(5.4.2)\quad = J_1-J_2, \text{ say.}$$

Since h has a compact support it follows as in the proof of estimate of J_3 in Theorem 5.3.1 that

$$(5.4.3)\quad |J_2| \leq \frac{M_1}{n^{(m+2)/2}} ||h||_{L_1[A,B]}.$$

Using Fubini's theorem we have

$$J_1 = \int_A^B \int_A^B W(n,t,u)\,F(t,u)g(t)\,du\,dt$$

$$= \int_A^B \int_A^B W(n,t,u)\,F(t,u)g(t)\,dt\,du.$$

Writing $h_r(u) = h(u)\, g^{(r)}(u)$, $u \in [A,B]$, where $0 \leq r \leq m$ as in the proof of Lemma 3.4.2 we have

$$J_1 = \sum_{r=0}^{m} \{ \frac{1}{r!} \{ \int_A^B \int_A^B W(n,t,u) \times$$

$$\times \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta^j((t-u)^r\, h_r(u))\} \, dt\, du\}\}$$

$$+ \frac{1}{(m+1)!} \{ \sum_{k=0}^{m} \{ \int_A^B \int_A^B W(n,t,u) a_k(t,u)(t-u-\frac{k}{n^{1/2}})^{m+1} \times$$

$$\times \{ g^{(m+1)}(\xi_k)\, h(u+\frac{k}{n^{1/2}})\} \, dt\, du\}\}$$

$$(5.4.4) \quad = \sum_{r=0}^{m+1} \frac{1}{r!} J_{1,r} \text{, say,}$$

where $a_k(t,u)$ is as defined in (3.4.3) and ξ_k lies between $u + \frac{k}{n^{1/2}}$ and t.

It follows from (3.4.3) and Corollary 1.8.10 that

$$(5.4.5) \qquad |J_{1,m+1}| \leq \frac{M_1'}{n^{(m+1)/2}} \, ||h||_{L_1[A,B]} .$$

Using Fubini's theorem we interchange the integrals in u and t in $J_{1,r}$ $(r = 1,2,\ldots,m)$ to get

$$(5.4.6) \qquad J_{1,r} = \int_A^B \bar{S}_{n,m}((t-u)^r\, h_r(u),t)\, dt.$$

We can write

$$(5.4.7) \quad h_r(u) = \sum_{k=0}^{m-r} \frac{(u-t)^k}{k!}\, h_r^{(k)}(t)$$

$$+ \frac{1}{(m-r)!} \int_t^u (u-w)^{m-r}\, h_r^{(m+1-r)}(w)\, dw.$$

It follows from (3.1.3), (5.3.5), (5.4.6) and (5.4.7) that

(5.4.8) $J_{1,r}$

$$= \frac{1}{(m-r)!} \int_A^B \bar{S}_{n,m}((t-u)^r \{\int_t^u (u-w)^{m-r} h_r^{(m+1-r)}(w)dw\},t)\,dt.$$

Since h_r has a compact support contained in (A,B), proceeding as in the proof of the estimate of $T_3(t)$ in Theorem 5.2.3 we obtain the estimate

$$(5.4.9) \quad |J_{1,r}| \le \frac{M_2}{n^{(m+1)/2}} ||h_r^{(m+1-r)}||_{L_1[A,B]}.$$

Using Lemma 1.2.2 this is further bounded as

$$(5.4.10) \quad |J_{1,r}| \le \frac{M_2'}{n^{(m+1)/2}} \{||h^{(m)}||_{L_1[A,B]} + ||h||_{L_1[A,B]}\}.$$

Now we evaluate $J_{1,0}$. From (5.4.4) we have

$$J_{1,0} = \int_A^B\int_A^B W(n,t,u)\,\{\sum_{j=0}^{m} \{\frac{n^{j/2}}{j!}(\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}}))\Delta^j h_o(u)\}\}\,dt\,du.$$

Writing $\prod_{i=0}^{j-1}(t-u-\frac{i}{n^{1/2}}) = (t-u)^j + \sum_{r=1}^{j-1} \frac{d_{j,r}}{n^{r/2}}(t-u)^{j-r}$,

for some constants $d_{j,r}$ it follows from an application of Corollary 1.8.9 and Lemma 3.4.3 that

$$J_{1,0} = \sum_{j=0}^{m} \{\frac{n^{j/2}}{j!}\{\int_A^B\int_A^B W(n,t,u)(t-u)^j\,\Delta^j\,h_o(u)dt\,du\}\}.$$

With $P_r(u,n) = P_r(u)$, defined as in Lemma 1.8.13,

$$J_{1,0} = \sum_{j=0}^{m} \{ \frac{n^{j/2}}{j!} \{ \int_A^B (\sum_{r=0}^{j} \binom{j}{r} P_r(u)(-u)^{j-r}) \Delta^j h_o(u)\, du \}\}.$$

By an application of Lemmas 1.8.12, 1.8.13 and 3.4.3, with $a_o = a(n)$ and $a_r = a(n)\{ \prod_{j=2}^{r+1} (1 - \frac{j\alpha}{n})^{-1}\}$ we obtain from above

$$J_{1,0} = \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\sum_{r=0}^{j} \binom{j}{r} (-1)^{j-r} a_r)(\int_A^B u^j \Delta^j h_o(u) du)$$

$$= (\int_A^B h_o(u)du) \{ \sum_{j=0}^{m} (-1)^j (\sum_{r=0}^{j} \binom{j}{r} (-1)^{j-r} a_r)\}$$

$$= (\int_A^B h_o(u)du) \{ \sum_{r=0}^{m} (-1)^r (\sum_{j=r}^{m} \binom{j}{r}) a_r \}$$

$$= (\int_A^B h_o(u)du) \{ \sum_{r=0}^{m} (-1)^r \binom{m+1}{r+1} a_r \}.$$

Next we put values of a_r from (1.8.11), to get

$$J_{1,0} = a(n)(\int_A^B h_o(u)du) \{ \sum_{r=0}^{m} (-1)^r \binom{m+1}{r+1} (\prod_{j=2}^{r+1} (1 - \frac{j\alpha}{n})^{-1})\}$$

(where $\prod_{j=2}^{r+1} (1 - \frac{j\alpha}{n})^{-1}$ for $r = 0$ is interpreted as 1)

$$= a(n)(1 - \frac{\alpha}{n})(\int_A^B h_o(u)du) \{ \sum_{r=0}^{m} (-1)^r \binom{m+1}{r+1}(\prod_{j=1}^{r+1} (1 - \frac{j\alpha}{n})^{-1})\}$$

$$= a(n)(1 - \frac{\alpha}{n})(\int_A^B h_o(u)du) \{ \sum_{r=1}^{m+1} (-1)^{r-1} \binom{m+1}{r}(\prod_{j=1}^{r} (1 - \frac{j\alpha}{n})^{-1})\}$$

$$= a(n)(1 - \frac{\alpha}{n})(\int_A^B h_o(u)du) \{ \sum_{r=0}^{m+1} (-1)^{r-1} \binom{m+1}{r} (\prod_{j=1}^{r} (1 - \frac{j\alpha}{n})^{-1})\}$$

$$+ a(n)(1 - \frac{\alpha}{n})(\int_A^B h_o(u)du).$$

Finally, we apply Corollary 3.4.5 to the right hand side of the above expression, to get

$$(5.4.11)\quad J_{1,0} = a(n)(1-\tfrac{\alpha}{n})(\int_A^B h_o(u)du)(1+O(\frac{1}{n^{[\frac{m}{2}]+1}})),\ (n\to\infty).$$

Since (5.4.11) holds for every choice of h and g satisfying the hypothesis of the lemma, we deduce from this the following :

$$(5.4.12)\qquad a(n)\,(1-\tfrac{\alpha}{n}) = 1 + O(\frac{1}{n^{(m+1)/2}}),\quad (n\to\infty).$$

To show this we choose $g \in C_o^{m+1}$ with supp $g \subset I_2^o$ and $g(t) = 1$ for $t \in I_3$. Next we choose $h \in C_o^{m+1}$ with supp $h \subset (A,B)$ and $h(t) = 1$ for $t \in I_1$. Then, with $\chi(t)$ as the characteristic function of I_2

$$J_{1,0} = \int_A^B \bar{S}_{n,m}(h_o,t)\,dt$$

$$= \int_A^B \chi(t)\,\bar{S}_{n,m}(h_o,t)dt + \int_A^B (1-\chi(t))\,\bar{S}_{n,m}(h_o,t)\,dt$$

$$(5.4.13)\quad = \Sigma_1 + \Sigma_2, \text{say}.$$

It is easy to see that for any fixed positive number ℓ

$$(5.4.14)\qquad \Sigma_2 = O(n^{-\ell}),\quad (n\to\infty).$$

By Theorem 5.2.6

$$\Sigma_1 = \int_{a_2}^{b_2} h_o(t)dt + O(\frac{1}{n^{(m+1)/2}}),\quad (n\to\infty),$$

$$(5.4.15)\qquad = \int_{a_2}^{b_2} g(t)dt + O(\frac{1}{n^{(m+1)/2}}),\quad (n\to\infty).$$

Thus (5.4.12) follows from (5.4.11) and estimate of $J_{1,0}$ given by (5.4.13), (5.4.14) and (5.4.15).

Finally from (5.4.11) and (5.4.12) we obtain

$$(5.4.16) \qquad J_{1,0} = (\int_A^B h_0(u)du)\,(1 + O(\frac{1}{n^{(m+1)/2}})), \quad (n \to \infty).$$

The lemma now follows from (5.4.2), (5.4.3), (5.4.4), (5.4.5), (5.4.10) and (5.4.16).

We may also note that the condition (1.5.9) is obtainable from (5.4.12) by taking limit as $n \to \infty$.

For the proofs of various saturation theorems in the thesis we have made use of the Euler-Maclaurin sum formula or the dual operators. Corresponding proofs of saturation theorems for convolution operators or operators for which a commuting approximation process is available are relatively simpler, as the mollifier technique becomes available (see e.g., [21] and [63]). In the case of convolutions the Fourier methods (see [20]) are also applicable. In the non-commutative case Micchelli [47] developed the use of semi-groups of operators (see [19]), applied through limits of certain iterates of operators, to obtain the saturation results (see also [59]).

Regarding our proofs of inverse theorems the approach is essentially motivated by the work on interpolation spaces even though explicitly we have not used the notion of K-functionals (see e.g., [24]) in the thesis.

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